Infinite Games
Lent Term 2021
Part III of the Mathematical Tripos
University of Cambridge
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Example Sheet #2

Examples Classes.
#2: Friday 26 February 2021, 3:30–5pm, Zoom.
#3: Friday 12 March 2021, 3:30–5pm, Zoom.
#4: Friday 30 April 2021, 3:30–5pm, Zoom.

Remote student interaction & presentations. As for the first example sheet, we ask you to arrange mathematical discussions about the material of *Infinite Games* in pairs: please arrange virtual meetings with one of the other students taking this course and work on Examples 15 & 18 together, preparing a brief online presentation of these examples for the second Examples Class. Your pair meetings can and should be about both the mathematical content of the two examples and about the practicalities of the presentation. (If you do not know how to find a partner for these meetings, do not hesitate to contact us by e-mail and we shall arrange some pairs.)

Marking. On the moodle page, there is an *Assignment* called *Example Sheet #2* in the section *Example Sheets & Classes*. You can submit your work as a single pdf file there. Feel free to submit all of your work; Examples 20 & 23 will be marked. Please submit your work by Thursday noon (i.e., **25 February 2021, 12pm GMT**).

(14) A function $c : \omega^{<\omega} \to \omega^{<\omega}$ is called *coherent* if for $p \subseteq q$, we have $c(p) \subseteq c(q)$, and for $x \in \omega^\omega$, we have $|c(x{\upharpoonright}n)| \to \infty$. If c is coherent, we define $f_c : \omega^\omega \to \omega^\omega$ by $f_c(x) := \bigcup_{n\in\omega} c(x{\upharpoonright}n)$. Prove that the following are equivalent:

(i) $f : \omega^\omega \to \omega^\omega$ is continuous and
(ii) there is a coherent c such that $f = f_c$.

(15) Presentation Example. Let $f : \omega^\omega \to \omega^\omega$ be any function and consider the following game $\mathrm{G}(f)$ on the move set $\omega \cup \{\mathsf{pass}\}$: player I may play only elements of ω, but player II may play pass; suppose player I produces x and player II produces a sequence $y \in (\omega \cup \{\mathsf{pass}\})^\omega$; remove all of the pass moves from y and obtain y^*; if $y^* \notin \omega^\omega$, then player II loses; otherwise player II wins if and only if $y^* = f(x)$. Show that the following are equivalent:

(i) $f : \omega^\omega \to \omega^\omega$ is continuous and
(ii) player II has a winning strategy in the game $\mathrm{G}(f)$.

What happens if you do not require the extra possibility of pass moves?

(16) Consider the real numbers $\mathbb{R}$ with their usual topology (i.e., not Baire space) and their subspace $\mathbb{Q}$. Show that $\mathbf{\Delta}^0_2(\mathbb{Q}) \neq \{A \cap \mathbb{Q}\,;\, A \in \mathbf{\Delta}^0_2(\mathbb{R})\}$.

(17) Again, consider the usual real numbers $\mathbb{R}$ with their usual topology and let $F \subseteq \mathbb{R}$ be closed. Show that for every continuous map $f : \mathbb{R} \to X$ where X is a metric space, the set $f[F]$ is $\mathbf{\Sigma}^0_2(X)$.

(18) Presentation Example. Let $n \in \omega$. Show that the sets $\mathbf{\Sigma}^1_n(\omega^\omega)$, $\mathbf{\Pi}^1_n(\omega^\omega)$, and $\mathbf{\Delta}^1_n(\omega^\omega)$ are closed under countable unions and intersections. Deduce that every Borel subset of ω^ω is in $\mathbf{\Delta}^1_1(\omega^\omega)$. In light of Example (17), your proof can't work on the usual real numbers. What goes wrong?

(19) Let $\mathbf{\Gamma}$ be a pointclass. We defined the concepts of being a *Y-universal set for $\mathbf{\Gamma}(X)$* and of being a $\mathbf{\Gamma}$-complete subset of X. What is the relationship between these two notions?

(20) Marked Examples. Let A and B be disjoint subsets of ω^ω. We say that A and B are *Borel separable* if there is a Borel set C such that $A \subseteq C$ and $B \cap C = \varnothing$.

Consider sets $\{A_n\,;\, n \in \omega\}$ and $\{B_n\,;\, n \in \omega\}$. Suppose that for each $n, m \in \omega$, A_n and B_m are Borel-separable. Then $\bigcup_{n\in\omega} A_n$ and $\bigcup_{n\in\omega} B_n$ are Borel separable.

(21) Show the *Luzin Separation Theorem*: any two disjoint analytic subsets of ω^ω are Borel separable. Deduce that a set $B \subseteq \omega^\omega$ is Borel if and only if it is $\mathbf{\Delta}^1_1(\omega^\omega)$.

(22) Let κ be the smallest cardinality such that there is some $A \subseteq \mathbb{R}$ with $|A| = \kappa$ which is not Lebesgue-null. Assume that A is such a set of cardinality κ and that R is a wellorder of A of order type κ. Show that R cannot be Lebesgue-measurable.

[*Hint.* Fubini's theorem in the following form may help: if $B \subseteq \mathbb{R}\times\mathbb{R}$ is Lebesgue-measurable, then it is a null set if and only if the set of all vertical (or horizontal) sections which are not null is null.]

(23) Marked Example. Let X be a topological space. A set $A \subseteq X$ is called *nowhere dense* if its closure does not contain a non-empty open set. It is called *meagre* if it is a countable union of nowhere dense sets.

Let $A \subseteq \omega^\omega$ and consider the following game $\mathrm{G}^{**}(A)$: players I and II play nonempty finite sequences $p_i \in \omega^{<\omega}$; consider $x := p_0p_1p_2\ldots$; player I wins if $x \in A$, otherwise player II wins. Show that

(i) Player I has a winning strategy in $\mathrm{G}^{**}(A)$ if and only if there is a position $p \in \omega^{<\omega}$ such that $[p]\backslash A$ is meagre.

(ii) Player II has a winning strategy in $\mathrm{G}^{**}(A)$ if and only if A is meagre.

(24) Let X be a set. If $R \subseteq X \times X$ is binary relation on X, we call it *serial* if for all $x \in X$ there is some $y \in X$ such that $x\ R\ y$. The following statement $\mathsf{DC}(X)$ is called the *principle of dependent choices for X*: for every serial relation R on X, there is some $f : \omega \to X$ such that for all $n \in \omega$, we have $f(n)\ R\ f(n+1)$.

Find a fragment of AC (in the sense of Example (4) on *Example Sheet #1*) that implies $\mathsf{DC}(X)$.

(25) Show that $\mathsf{DC}(X^{<\omega})$ is equivalent to the statement "for every tree T on X, the structure $(T, \supseteq)$ is wellfounded if and only if $[T] = \varnothing$".