Example Sheet #1

Examples Classes.
#1. Friday 12 February 2021, 3:30–5pm, Zoom.
#2: Friday 26 February 2021, 3:30–5pm, Zoom.
#3: Friday 12 March 2021, 3:30–5pm, Zoom.
#4: Friday 30 April 2021, 3:30–5pm, Zoom.

Remote student interaction & presentations. In the new world of online university teaching, many of us miss the informal discussions about mathematics that we used to have at the CMS. Chatting before or after lectures, listening to someone else speaking at the blackboard, chance conversations during lunch or dinner are all missing in our current university experience.

In an attempt to encourage the creation of virtual informal discussions, we ask you to arrange mathematical discussions about the material of *Infinite Games* in pairs: please arrange virtual meetings with one of the other students taking this course and work on Examples 11 & 12 together, preparing a brief online presentation of these examples for the first Examples Class. (If you do not know how to find a partner for these meetings, do not hesitate to contact us by e-mail and we shall arrange some pairs.)

A positive side effect is that these meetings allow you to discuss online presentations of mathematics with your fellow students. We expect that online presentations will be a crucial skill for many of us in the future and getting experience with them is very important.

Mathematical online presentations would be done either by sharing the screen (e.g., the Zoom whiteboard or some other writing software) and writing on it (e.g., with a pen on a tablet) or by writing on a piece of paper with a camera set up to point to the paper (i.e., a visualiser). Your pair meetings can and should be about both the mathematical content of the two examples and about the practicalities of the presentation.

The Examples Class will then start with student presentations of Examples 11 & 12. However, active student presentation of examples is not restricted to these two examples: we greatly encourage if students present their solutions (or partial solutions, solution ideas, failed attempts of solutions etc.) to all examples during the Examples Class.

Marking. On the moodle page, there is an *Assignment* called *Example Sheet #1* in the section *Example Sheets & Classes*. You can submit your work as a single pdf file there. Feel free to submit all of your work; examples 6 & 10 will be marked. Please submit your work by Thursday noon (i.e., **11 February 2021, 12pm GMT**).

Notation. In the following, M is the set of possible moves, $M^{<\omega}$ the set of positions, and M^ω the set of plays. If $s \in M^{<\omega}$ and $x \in M^\omega$, we write sx for the concatenation of s and x, i.e., $sx(n) := s(n)$ for $n < |s|$ and $sx(n) := x(n - |s|)$ for $n \geq |s|$. If $m \in M$, we write mx for the concatenation of the length one sequence with element m and the infinite sequence x.

(1) If $x \in M^\omega$, we defined strategies σ_x and τ_x as the strategies for player I or player II, respectively, that play x independently of what the other player is doing. These strategies were called *blindfolded*. Show that a strategy σ is a winning strategy for player I in the game $\mathrm{G}(A)$ if and only if it wins against all blindfolded strategies for player II (i.e., $\sigma * \tau_x \in A$ for all $x \in M^\omega$).

(2) If $A, B \subseteq M^\omega$ are disjoint, we define the winning conditions for the game $\mathrm{G}_{\mathrm{draw}}(A,B)$ as follows: if the play of the game is $x \in M^\omega$, then player I wins if $x \in A$, player II wins if $x \in B$, and otherwise the game is a *draw*. A strategy σ is *winning* or *(at least) drawing in* $\mathrm{G}_{\mathrm{draw}}(A,B)$ *for player I* if for all strategies τ, we have that $\sigma * \tau \in A$ or $\sigma * \tau \notin B$, respectively. A strategy τ is *winning* or *(at least) drawing in* $\mathrm{G}_{\mathrm{draw}}(A,B)$ *for player II* if for all strategies σ, we have that $\sigma * \tau \in B$ or $\sigma * \tau \notin A$, respectively.

Show that if A and $M^\omega \backslash B$ are determined, then at least one of the two players has a drawing strategy in $\mathrm{G}_{\mathrm{draw}}(A,B)$ and that both players have a drawing strategy if and only if none of them has a winning strategy.

(3) Let $\mu : M^{<\omega} \to \{\mathrm{I}, \mathrm{II}\}$ be a function; we call it a *move function*. Let $A \subseteq M^\omega$ and consider the game $\mathrm{G}_\mu(A)$ defined as follows: if the game has reached position $s \in M^{<\omega}$, then player $\mu(s)$ makes the next move and if $x \in M^\omega$ is the run of the game, then player I wins if and only if $x \in A$. Write $\mathrm{G}_\mu(A)$ as a game with rules $\mathrm{G}(A;T)$ as defined in Lecture II (pp. 7 & 8 of the lecture notes).

(4) Let X and Y be sets. By $\mathsf{AC}_Y(X)$ we denote the *axiom of choice for Y-parametrised families of subsets of X*, i.e., the statement that for each family $\{A_y\,; y \in Y\}$ of non-empty subsets of X there is a choice function $c : Y \to X$ with $c(y) \in A_y$ for all $y \in Y$.

Show (in ZF without the axiom of choice) that $\mathsf{AC}_\omega(\mathbb{R})$ implies that $\aleph_1$ is a regular cardinal.

(5) Let X be a set and $A \subseteq X \times X$. A partial function $f : X \to X$ is called a *uniformisation of A* if for each $x \in X$, if there is some $y \in X$ such that $(x,y) \in A$, then $x \in \mathrm{dom}(f)$ and for all $x \in \mathrm{dom}(f)$, we have that $(x, f(x)) \in A$. We say that X *satisfies the uniformisation principle* if each subset $A \subseteq X \times X$ has a uniformisation.

Show (in ZF without the axiom of choice) that $\mathsf{AC}_X(X)$ implies that X has the uniformisation principle.

(6) [Marked example] Show (in ZF without the axiom of choice) that the determinacy of all finite games on a set of moves M implies $\mathsf{AC}_M(M)$.

(7) Show (in ZF without the axiom of choice) that the statement "every quasi-determined subset of M^ω is determined" is equivalent to $\mathsf{AC}_{M^{<\omega}}(M)$.

(8) Show that a tree Q is a quasi-strategic tree for player I if and only if it is the union of strategic trees for player I (and similarly for player II). Discuss how much of the axiom of choice your proof needed. Is it possible that the proof works in ZF?

(9) Construct a set $A \subseteq M^\omega$ such that player I has a winning strategy in both $\mathrm{G}(A)$ and $\mathrm{G}(M^\omega \backslash A)$.

(10) [Marked example] If $A \subseteq M^\omega$ and $m \in M$, let $A_m := \{x \in M^\omega\,; mx \in A\}$. Assume that for every $m \in M$, the set A_m is determined. Prove that $M^\omega \backslash A$ is determined.

(11) [Presentation example] Assume AC and show that

(a) there is a determined set whose complement is not determined,
(b) there are determined sets whose intersection is not determined, and
(c) there are determined sets whose union is not determined.

(12) [Presentation example] In the proof of the Gale-Stewart theorem, the winning strategy for player I was "stay on positions with your label" and the winning strategy for player II was "stay on positions with your label and reduce the age of the position". Show that, in general, "stay on positions with your label" is not enough for player II in the Gale-Stewart proof, but that if the payoff set A is clopen (i.e., both A and its complement are open), then it is enough.

(13) Let $A \subseteq \omega^\omega$ be the difference of two open sets (i.e., there are open sets P and Q such that $A = P \backslash Q$). Show that A is determined.