Infinite Games
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Part III of the Mathematical Tripos
University of Cambridge
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Twenty-fourth Lecture

The material of the twenty-fourth lecture is non-examinable.

In previous lectures, we have proved the following results under AD:

(1) AD implies $\mathsf{AC}_\omega(\mathbb{R})$ (Twenty-first Lecture);
(2) $\mathsf{AC}_\omega(\mathbb{R})$ implies that $\aleph_1$ is regular (Example # 6 on Example Sheet # 1);
(3) AD implies that every ultrafilter is $\aleph_1$-complete (Twenty-first & Twenty-second Lecture);
(4) AD implies that there is a non-principal, $\aleph_1$-complete ultrafilter on $\aleph_1$ (Twenty-third lecture);
(5) AD implies that there is a normal, non-principal, $\aleph_1$-complete ultrafilter on $\aleph_1$, i.e., $\aleph_1$ is measurable (Example # 54 on Example Sheet # 4).

Some of our proofs depended on unproved results from the research areas of *inner model theory* about the cardinal structure in models of the form $\mathbf{L}(x)$. In the following, we shall discuss further structural consequences for cardinals under the assumption of AD. Many of these results use non-trivial results from both *inner model theory* and the *theory of forcing*, so we can only give proofs for a few results and otherwise proof sketches, proof ideas, or pointers to the literature.

The *Axiom of Dependent Choice* DC is the statement: for any set X and any binary relation on X with the property $\forall x \exists y (x \, R \, y)$ there is a function $f : \omega \to X$ such that for all $n \in \omega$, we have that $f(n) \, R \, f(n+1)$; the *Axiom of Countable Choice* AC_ω is the statement that for all sets X, $\mathsf{AC}_\omega(X)$ holds.

Proposition 1. DC *implies* AC_ω.

P r o o f. Let $\mathcal{A} := \{A_n \, ; \, n \in \omega\}$ be a family of non-empty subsets of X, without loss of generality, pairwise disjoint. Define $R \subseteq X \times X$ by $x \, R \, y$ if and only if there is some n such that $x \in A_n$ and $y \in A_{n+1}$. The sequence given by DC is a choice function for $\mathcal{A}$. Q.E.D

Proposition 2. AC_ω *implies that* $\mathrm{cf}(\kappa^+) > \aleph_0$ *for all cardinals* κ.

P r o o f. This is just an inspection of the usual regularity proof of κ^+ under AC: suppose $\kappa^+ = \bigcup_{n\in\omega} \kappa_n$ for some $\kappa_n < \kappa^+$. Then for each n, the set of surjections from κ onto κ_n is non-empty, so by AC_ω, we can pick one, say, $\pi_n : \kappa \twoheadrightarrow \kappa_n$. Now $(\alpha, n) \mapsto \pi_n(\alpha)$ is a surjection from $\kappa \times \aleph_0$ onto κ^+. Contradiction! Q.E.D

§ 1. What does AD say about $\aleph_2$? In our analysis of $\aleph_1$, we used the fact that there is a close connection between elements of $\aleph_1$ (i.e., countable ordinals) and elements of Baire space. Elements of $\aleph_2$ are in general coded by functions from $\aleph_1$ to 2, but these can be captured by elements of Baire space as follows:

Lemma 3 (AD; Solovay)**.** *Every function* $u : \aleph_1 \to 2$ *there is an* $x \in \omega^\omega$ *such that* $u \in \mathbf{L}(x)$.

[This is an argument combining a Solovay game similar to those from Examples # 33 & # 54 and a forcing argument; for details, cf. [3, Theorem 28.5].]

Note that Lemma 3 is a special case of a more general phenomenon known as the *Moschovakis Coding Lemma*: AD implies that if there is a surjection from ω^ω to some cardinal κ, then there is a surjection from ω^ω to the set κ^κ of functions from κ to κ [3, Theorem 28.15].

We can now use Lemma 3 to lift the proof of (4) to $\aleph_2$:

Theorem 4 (AD; Solovay)**.** *The cardinal* $\aleph_2$ *has an* $\aleph_2$*-complete non-principal ultrafilter. In particular,* $\aleph_2$ *is regular.*

P r o o f. In the proof of (4), we used the Martin measure $\mathrm{M_D}$ on $\mathcal{D}_\mathrm{D} := \omega^\omega/{\equiv_\mathrm{D}}$. We used the function $f : \mathcal{D}_\mathrm{D} \to \aleph_1 : x \mapsto \aleph_1^{\mathbf{L}(x)}$ to get a non-principal ultrafilter on $\aleph_1$.

Using the same idea, but the function $g : \mathcal{D}_\mathrm{D} \to \aleph_2$ mapping x to the $\mathbf{L}(x)$-cardinal successor of $\aleph_1^\mathbf{V}$ (of course, one needs to prove that this is an ordinal $< \aleph_2$), we get an ultrafilter $U := \{A \subseteq \aleph_2\,;\, g^{-1}[A] \in \mathrm{M_D}\}$. The ultrafilter U is non-principal by Lemma 3: suppose that $\{\alpha\} \in U$, then $\alpha < \aleph_2$, so find a function $u : \aleph_1 \to 2$ encoding α as a subset of $\aleph_1$ and find $x \in \omega^\omega$ such that $u \in \mathbf{L}(x)$. But then for each $z \geq_\mathrm{D} x$, $g(x) > \alpha$, so $g^{-1}(\{\alpha\})$ cannot contain a cone.

The $\aleph_1$-completeness of U comes for free from (3), but in order to prove that it is $\aleph_2$-complete, we need to use another combination of a Solovay game and a forcing argument very similar to the proof of Lemma 1 [3, Lemma 28.6]. Q.E.D

§ 2. The projective ordinals. Just under the assumption of PD (the determinacy of all projective sets), a rather extensive structure theory for the projective sets had been developed in the late 1960s and early 1970s by Addison, Martin, Moschovakis, and Kechris. The central concepts here are structural properties of pointclasses known as the *prewellordering property*, the *scale property*, the *uniformisation property*, the *reduction property*, and the *separation property*. The uniformisation property is a definable version of the *uniformisation principle* from Example # 7 on Example Sheet # 1 and the scale property is a definable version of being *Suslin*. The main results in this projective structure theory are the so-called *Periodicity Theorems* under the assumption of $\mathsf{ZFC}+\mathsf{PD}$: the first Periodicity Theorem (Addison-Martin-Moschovakis) states that the prewellordering property propagates from the pointclass $\mathbf{\Sigma}^1_n$ to $\mathbf{\Pi}^1_{n+1}$ [3, Theorem 29.13] and the second Periodicity Theorem (Moschovakis) states the same for the scale property [3, Theorem 30.8]. As a consequence of the second Periodicity Theorem, we get tree representations for all projective sets and can even say something about the size of the trees representing them:

A relation $\leq\, \subseteq \omega^\omega \times \omega^\omega$ is called a *prewellorder* if it is reflexive, transitive, total, and well-founded. As usual, if $\leq$ is a prewellorder, then $\equiv$, defined by $x \equiv y :\iff x \leq y$ and $y \leq x$, is an equivalence relation and $\leq$ defines a wellorder on $\omega^\omega/{\equiv}$. The *length* of $\leq$, denoted by $\|{\leq}\|$, is the order type of $(\omega^\omega/{\equiv}, \leq)$. We then define the *nth projective ordinal* to be

$$\boldsymbol{\delta}^1_n := \sup\{\alpha\,;\,\text{there is a } \mathbf{\Delta}^1_n \text{ prewellorder} \leq \text{with } \|{\leq}\| = \alpha\}.$$

Proposition 5 (ZF)**.** $\boldsymbol{\delta}^1_1 = \aleph_1$.

P r o o f. If $\leq$ is a $\mathbf{\Delta}^1_1$ prewellorder, then the set of codes of $\leq$ is a $\mathbf{\Sigma}^1_1$ subset of WF, thus by the Boundedness Lemma, bounded by $\alpha < \aleph_1$. But that means that $\|{\leq}\| \leq \alpha < \aleph_1$. Q.E.D

The second Periodity Theorem implies that

Corollary 6 ($\mathsf{DC}+\mathsf{PD}$; cf., e.g., [3, Exercise 30.12])**.** *Every $\mathbf{\Pi}^1_n$-set is $\boldsymbol{\delta}^1_n$-Suslin.*

Note that in light of Proposition 5, Shoenfield's Theorem (from the Nineteenth Lecture) is the case $n = 1$ of this result. We end this section by giving to general results about Suslin sets from descriptive set theory:

Proposition 7 (ZFC; Martin)**.** *If A is $\aleph_n$-Suslin, then it is the union of $\aleph_n$ many Borel sets.*

P r o o f. We have already proved the case $n = 1$ as our general analysis of $\mathbf{\Pi}^1_1$ sets. We shall show the result by induction on n. Suppose $A = p[T]$ is $\aleph_n$-Suslin, i.e., T is a tree on $\aleph_n \times \omega$. For each $\gamma < \aleph_n$, define

$$T^\gamma := T \cap (\gamma \times \omega)^{<\omega}.$$

We claim that $A = \bigcup_{\gamma<\aleph_n} p[T^\gamma]$. The inclusion from right to left is obvious, so let $x \in A$, i.e., there is some $u : \omega \to \aleph_n$ such that $(u, x) \in [T]$. But then let $\gamma := \sup\{u(n)\,;\, n \in \omega\} < \aleph_n$ (by regularity of $\aleph_n$) and observe that $u : \omega \to \gamma$, so $(u, x) \in [T^\gamma]$.

Clearly, $p[T^\gamma]$ is γ-Suslin, but this means that it is $\aleph_{n-1}$-Suslin. We use the induction hypothesis to get that it is a union of $\aleph_{n-1}$ many Borel sets. But now A is a union of $\aleph_n \cdot \aleph_{n-1} = \aleph_n$ many Borel sets. Q.E.D

Theorem 8 (Kunen-Martin Theorem; DC; [4, Theorem 3.21])**.** *If $\leq$ is a prewellorder on ω^ω that is κ-Suslin, then $\|{\leq}\| < \kappa^+$.*

Note that the Kunen-Martin Theorem in combination with Shoenfield's theorem implies that $\boldsymbol{\delta}^1_2 \leq \aleph_2$.

§ 3. The projective ordinals under $\mathsf{AD}+\mathsf{DC}$. Since Corollary 6 is proved in the theory $\mathsf{ZF}+\mathsf{DC}+\mathsf{PD}$, it is true in models of $\mathsf{AD}+\mathsf{DC}$, so it makes sense to ask what we know about the projective ordinals under these conditions. Using (a more specific statement of) the mentioned Moschovakis Coding Lemma, one obtains [4, Theorem 2.5 & 4.1]:

Proposition 9 ($\mathsf{AD}+\mathsf{DC}$; Moschovakis)**.** *All projective ordinals are distinct regular cardinals.*

In combination with Theorem 8, this allows us to determine $\boldsymbol{\delta}^1_2 = \aleph_2$. The analysis behind Proposition 9 gives us more information about the projective ordinals: the odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$ are always successors of cardinals of cofinality $\aleph_0$ [4, Theorem 3.20] and the even projective ordinals are their successors, i.e., $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$ [4, Theorem 3.22]. In light of Proposition 2, this implies that $\boldsymbol{\delta}^1_3 \geq \aleph_{\omega+1}$ (since $\aleph_\omega$ is the smallest cardinal of cofinality $\aleph_0$ bigger than $\aleph_2$).

The general analysis of the projective ordinals under AD was a joint effort of many authors that took several years: Kunen and Martin had shown in the early 1970s that all projective ordinals are measurable [4, Theorem 5.1] and Martin's analysis of $\boldsymbol{\Sigma}^1_3$ sets had shows that the mentioned lower bound for $\boldsymbol{\delta}^1_3$ is sharp: $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$ and $\boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$.

The calculation of the projective ordinals in general was done by Jackson in 1985 and required a much more careful look at all cardinals: all cardinals between $\boldsymbol{\delta}^1_1$ and $\boldsymbol{\delta}^1_3$, i.e., the $\aleph_n$ for $n \geq 2$ are represented as *iterated ultrapowers* of $\aleph_1$ with its normal ultrafilter $\mathcal{C}$ (cf. Example # 54 on Example Sheet # 4); this pattern transfers to arbitrary odd projective ordinals, so every cardinal between $\boldsymbol{\delta}^1_{2n+1}$ and $\boldsymbol{\delta}^1_{2n+3}$ can be described as an iterated ultrapower of $\boldsymbol{\delta}^1_{2n+1}$ with an ultrafilter derived from the normal measures on it. As a consequence, the calculation of $\boldsymbol{\delta}^1_{2n+3}$ can be reduced to the analysis of the normal measures on $\boldsymbol{\delta}^1_{2n+1}$ which in turn can be reduced to counting the number of regular cardinals below $\boldsymbol{\delta}^1_{2n+1}$.

Jackson's results can be summarised in the following theorem:

Theorem 10 ($\mathsf{AD}+\mathsf{DC}$; Jackson)**.** *Define $E(0) := 1$ and $E(n+1) := \omega^{E(n)}$. There are 2^n+1 many regular cardinals below $\boldsymbol{\delta}^1_{2n+3}$ and $\boldsymbol{\delta}^1_{2n+3} = \aleph_{E(2n+1)+1}$.*

[This is a famous theorem that is nowhere published in full. Jackson published a book in which he calculated $\boldsymbol{\delta}^1_5$ in detail [1]. The general argument is inductive: once $\boldsymbol{\delta}^1_{2n+1}$ is determined and has 2^n-1 regular cardinals below it, consider the normal measures

$$\mathcal{C}^\kappa_{\boldsymbol{\delta}^1_{2n+1}} := \{A \subseteq \boldsymbol{\delta}^1_{2n+1}\,;\text{ there is a club set } C \text{ such that } C \cap \{\alpha\,;\,\mathrm{cf}(\alpha)=\kappa\} \subseteq A\}$$

and show that every cardinal between $\boldsymbol{\delta}^1_{2n+1}$ and $\boldsymbol{\delta}^1_{2n+3}$ is an ultrapower by an ultrafilter expressible in terms of these measures and operations $\oplus$ and $\otimes$. As a consequence, the order type of the set of cardinals between $\boldsymbol{\delta}^1_{2n+1}$ and $\boldsymbol{\delta}^1_{2n+3}$ is the order type of the corresponding *ordinal algebra*. For details of the combinatorics of calculation, cf. [2].]

§ 4. Consequences for the ZFC-context. The results of § 3 may come across as rather esoteric: models of AD are models in which the majority of successor cardinals are singular with some weirdly complicated pattern of cofinalities. As Kanamori puts it: "It is a brave new world that has these properties! [3, p. 389]". However, the fact that large cardinals imply the existence of models of AD allows us to use this even in the ZFC-context.

In the Twenty-first Lecture, we discussed the Martin-Steel theorem that obtains PD from large cardinals. The following theorem by Woodin is a strengthening of it. Here $\mathbf{L}(\mathbb{R})$ is the smallest transitive model of ZF that contains the set $\mathbb{R}$ of all reals.

Theorem 11 (ZFC; Woodin; [3, Theorem 32.9])**.** *If there are infinitely many Woodin cardinals and a measurable above them all, then $\mathbf{L}(\mathbb{R}) \models \mathsf{AD}+\mathsf{DC}$.*

We use this to get the following theorem about the cardinality of projective sets. Compare this with Example # 29 from Example Sheet # 3: an uncountable $\boldsymbol{\Pi}^1_1$ set can have either cardinality $\aleph_1$ or $2^{\aleph_0}$. Observe also that the result does not mention infinite games or determinacy at all, but is a statement about sizes of definable sets under the assumption of the existence of large cardinals: the infinite games are hidden in the proof.

Theorem 12. *If there are infinitely many Woodin cardinals and a measurable cardinal above them all and if A is a projective set, then the cardinality of A is either $2^{\aleph_0}$ or $< \aleph_\omega$.*

P r o o f. First of all, by the Martin-Steel theorem, the assumption implies PD, so the set A is $\boldsymbol{\delta}^1_n$-Suslin for some n by Corollary 6.

By Theorem 11, we know that $\mathbf{L}(\mathbb{R}) \models \mathsf{AD} + \mathsf{DC}$, so the theory of § 3 applies. Thus, we know that in $\mathbf{L}(\mathbb{R})$, $\boldsymbol{\delta}^1_{2n+3}$ is the $(2^n + 1)$st regular cardinal. Since regularity of cardinals is downwards absolute and in the ZFC-universe, $\aleph_{2^n+1}$ is the $(2^n + 1)$st regular cardinal, we know that

$$(*) \qquad (\boldsymbol{\delta}^1_{2n+3})^{\mathbf{L}(\mathbb{R})} \leq \aleph_{2^n+1}.$$

Since $\mathbf{L}(\mathbb{R})$ contains all reals, all first-order statements about $\mathbb{R}$ are absolute between $\mathbf{L}(\mathbb{R})$ and the universe. In particular, $\mathbf{L}(\mathbb{R})$ computes all projective ordinals correctly, so

$$(\dagger) \qquad (\boldsymbol{\delta}^1_{2n+3})^{\mathbf{L}(\mathbb{R})} = \boldsymbol{\delta}^1_{2n+3}.$$

But (*) and (†) together mean that A is $\aleph_n$-Suslin for some $n < \omega$. By Proposition 7, we know that this means that A is a union of $\aleph_n$ many Borel sets. But Borel sets have the perfect set property, so if none of them has cardinality $2^{\aleph_0}$, then all of them are countable, but then $|A| \leq \aleph_n$. Q.E.D

References

[1] S. Jackson, *A Computation of $\boldsymbol{\delta}^1_5$*, Memoirs of the American Mathematical Society, Vol. 140 (American Mathematical Society, 1999).

[2] S. Jackson, B. Löwe, Canonical Measure Assignments, J. Symb. Log. 78 (2013): 403–424.

[3] A. Kanamori, *The Higher Infinite. Large Cardinals in Set Theory from Their Beginnings*, Perspectives in Mathematical Logic (Springer-Verlag 1994).

[4] A. S. Kechris, AD and projective ordinals, in: *Wadge Degrees and Projective Ordinals: The Cabal Seminar, Volume II*, edited by A. S. Kechris, B. Löwe, J. R. Steel, Lecture Notes in Logic. Vol. 37 (Cambridge University Press, 2011), pp. 304–345.