Infinite Games
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Example (34)

Unfortunately, the roles of players I and II in Examples (33) and (34) were flipped, resulting in a false statement for Example (33) and an almost trivial statement for Example (34). The incorrect statement for Example (33) was corrected, but the corresponding statement for Example (34) was not. The original statement of Example (34) read:

> Let σ be a winning strategy for player I in some Solovay game $\mathrm{G_S}(A)$. Show that there is a function $f : \aleph_1 \to \aleph_1$ such that if $\alpha = \|y\| < \xi$, then there is some x such that $\|x\| < f(\xi)$ and $(x, y) \in A$.

In the proof of Example (33), we established that in every Solovay game and for every winning strategy σ for player I, the set $\{x\,;$ there is a y such that x is the result of playing σ against $y\}$ is a $\mathbf{\Sigma}^1_1$ subset of WO, thus bounded by some $\alpha < \aleph_1$. So, the constant function $f(\xi) := \alpha + 1$ is an answer to the above formulation of Example (34).

The intention of Example (34) was the following statement:

> Let τ be a winning strategy for player II in some Solovay game $\mathrm{G_S}(A)$. Show that there is a function $f : \aleph_1 \to \aleph_1$ such that if $\|x\| < \xi$, then there is some y such that $\|y\| < f(\xi)$ and $(x, y) \notin A$.

The set we are considering now is $Y_\xi := \{y\,; \exists x \in \mathrm{WO}_{<\xi}(y = (\sigma_x * \tau)_{\mathrm{II}})\}$. Since $\mathrm{WO}_{<\xi}$ is a Borel set, we observe that Y_ξ is $\mathbf{\Sigma}^1_1$. Since τ was a winning strategy, we observe that playing τ against an element of WO will produce an element of WO. Thus, $Y_\xi \subseteq \mathrm{WO}$, so the boundedness lemma applies, and we find α_ξ such that $Y_\xi \subseteq \mathrm{WO}_{\leq\alpha_\xi}$. Thus, the function $f : \xi \mapsto \alpha_\xi + 1$ is the desired function.