Infinite Games
Lent Term 2020
Part III of the Mathematical Tripos
University of Cambridge
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Example Sheet #4

Examples Classes.
#1. Monday 3 February 2020, 3:30-5pm, MR14.
#2: Monday 17 February 2020, 5-6:30pm, MR14.
#3: Monday 2 March 2020, 3:30-5pm, MR14.
#4: Friday 1 May 2020, 1:30–3pm, MR4.
Revision: Friday 22 May 2020, 1:30–3pm, MR5.
You hand in your work at the beginning of the Examples Class.

42. In the proof of Shoenfield's Theorem (cf. the pdf file in the description of the *Nineteenth Lecture* website), we used "order preserving codes". Rewrite the proof to use Kleene-Brouwer codes instead.

43. Suppose that $M \subseteq \mathbf{V}$ is a transitive model such that M contains all elements of ω^ω. Show that if $\mathbf{V} \models \mathsf{ZF} + \mathsf{AD}$, then $M \models \mathsf{ZF} + \mathsf{AD}$ and $\aleph_1^M = \aleph_1$.

44. Show in ZF that AD_2 is equivalent to AD_ω.

45. Show in ZF that AD implies that there is no injection from $\aleph_1$ into ω^ω.

46. Assume ZF+"there is an injection from $\aleph_1$ into ω^ω". Show that ω^ω is a disjoint union of $\aleph_1$ many $\mathbf{\Pi}^0_3$ sets.

47. Let X be a non-empty set. A set $C \subseteq X^\omega$ is *closed* if there is a tree T on X such that $C = [T]$. Show in ZF, that "for every closed $C \subseteq X^\omega$, the game $\mathrm{G}(C)$ is determined" is equivalent to $\mathsf{AC}_X(X)$.

48. Let $\mathcal{D}$ be the collection of determined sets. Show in ZF that AD is equivalent to the statement "$\mathcal{D}$ is a σ-algebra".

49. In the lectures, we introduced the relation $\leq_{\mathrm{D}}$ on $\mathcal{R} = (\mathbf{V}_{\omega+1}, \in)$ defined by $x \leq_{\mathrm{D}} y$ if any only if there is a formula φ in two free variables such that "$\mathcal{R} \models \varphi(v, w)$" is absolute for transitive models of set theory and

$$z \in x \iff \mathcal{R} \models \varphi(z, y).$$

Show that $\leq_{\mathrm{D}}$ is a partial preorder (i.e., reflexive and transitive).

50. Let U be an $\aleph_1$-complete ultrafilter on X and $F : X \to \omega^\omega$. Show that F is constant on a set in U.

51. Consider the set $\mathcal{D}_\mathrm{D} := \omega^\omega/{\equiv_\mathrm{D}}$ and show that AD implies that there is no injection from $\mathcal{D}_\mathrm{D}$ into ω^ω.

52. In this example, use the mentioned properties of Gödel's *real model family* $\vec{\mathbf{L}} = \{\mathbf{L}(x)\,;\, x \in \omega^\omega\}$. In general, $\aleph_1$ is a cardinal in each of the $\mathbf{L}(x)$, so for each x there is some α_x such that $\aleph_1 = \aleph_{\alpha_x}^{\mathbf{L}(x)}$. Assume that $\aleph_1$ is inaccessible by reals for $\vec{\mathbf{L}}$ and determine α_x.

53. A subset $C \subseteq \aleph_1$ is called *closed* if for each $\gamma \in \aleph_1$, if $C \cap \gamma$ is unbounded in γ, then $\gamma \in C$; it is called *unbounded* if for each $\alpha \in \aleph_1$, there is a $\gamma \in C$ such that $\alpha < \gamma$; it is called *club* if it is closed and unbounded. The set $\mathcal{C} := \{A \subseteq \aleph_1\,;$ there is a club set C with $C \subseteq A\}$ is called the *club filter* on $\aleph_1$.

    Show that it is a non-principal, normal filter on $\aleph_1$.

54. If $A \subseteq \aleph_1$, we define the following *Solovay game* $\mathrm{G_S}(A)$: players I and II play in alternation; player I produces $x \in \omega^\omega$ and player II produces $y \in \omega^\omega$. Consider the functions $\{(x)_n\,;\, n \in \omega\} \cup \{(y)_n\,;\, n \in \omega\} \subseteq \omega^\omega$. If one of them is not in WO, then player II loses if the least m such that $(y)_m \notin \mathrm{WO}$ is smaller than the least m such that $(x)_m \notin \mathrm{WO}$; otherwise player I loses. If all of them are in WO, define

    $$\gamma := \sup(\{\|(x)_n\|\,;\, n \in \omega\} \cup \{\|(y)_n\|\,;\, n \in \omega\})$$

    and say that player I wins if $\gamma \in A$.

    Prove: if player I has a winning strategy in $\mathrm{G_S}(A)$, then $A \in \mathcal{C}$; player I has a winning strategy in $\mathrm{G_S}(A)$, then $\aleph_1 \backslash A \in \mathcal{C}$.

    Deduce that AD implies that $\aleph_1$ is measurable.