Topics in Set Theory
Lent Term 2018
Part III of the Mathematical Tripos
University of Cambridge
Prof. Dr. B. Löwe

Example Sheet #4

Example Classes.
#1. Thursday 1 February, 3–4pm, MR13.
#2. Thursday 15 February, 3–4pm, MR20.
#3. Thursday 1 March, 3–4pm, MR21.
#4. Wednesday 14 March, 3–4pm, MR21.
You hand in your work at the beginning of the Example Class.

(35) Let G be $\mathbb{P}$-generic over M and let $X \in M[G]$ be a family of non-empty sets. Explicitly define a name τ and prove that $f := \mathrm{val}(\tau, G)$ is a choice function for X, i.e., $\mathrm{dom}(f) = X$ and $f(x) \in x$ for any $x \in X$.

(36) If $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbf{1}_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}})$ are partial orders, then a function $i : \mathbb{P} \to \mathbb{Q}$ is called a *complete embedding* if

(a) i is order preserving, i.e., if $p \leq_{\mathbb{P}} p'$, then $i(p) \leq_{\mathbb{Q}} i(p')$;

(b) i preserves incompatibility in both directions, i.e., $p \perp_{\mathbb{P}} p'$ if and only if $i(p) \perp_{\mathbb{Q}} i(p')$; and

(c) for all $q \in \mathbb{Q}$ there is a $p \in \mathbb{P}$ such that for all $p' \leq_{\mathbb{P}}$, we have that $i(p')$ and q are compatible in $\mathbb{Q}$.

Suppose that $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding with $i, \mathbb{P}, \mathbb{Q} \in M$ and let H be $\mathbb{Q}$-generic over M. Show that $G := \{p \in \mathbb{P}\,;\, i(p) \in H\}$ is $\mathbb{P}$-generic over M and that $M[G] \subseteq M[H]$.

(37) If $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbf{1}_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbf{1}_{\mathbb{Q}})$ are partial orders, then a function $i : \mathbb{P} \to \mathbb{Q}$ is called a *dense embedding* if

(a) i is order preserving, i.e., if $p \leq_{\mathbb{P}} p'$, then $i(p) \leq_{\mathbb{Q}} i(p')$;

(b) i preserves incompatibility, i.e., if $p \perp_{\mathbb{P}} p'$, then $i(p) \perp_{\mathbb{Q}} i(p')$; and

(c) the image of $\mathbb{P}$ under i is dense in $\mathbb{Q}$.

Show that every dense embedding is a complete embedding.

(38) A family of finite sets $\mathcal{D}$ is called a Δ-*system* if there is a finite set R (called the *root of the* Δ-*system*) such that for all $D, D' \in \mathcal{D}$, if $D \neq D'$, then $D \cap D' = R$. Show that any uncountable family of finite sets contains an uncountable Δ-system. (*Hint.* Argue that you can assume w.l.o.g. that all elements of the family have the same size and prove the claim by induction on the size of the elements of the family.)

In the following, we assume that M is a countable transitive model of ZFC.

(39) We say that $\mathbb{P}$ *preserves cofinalities* if for every $\mathbb{P}$-generic filter G over M and every limit ordinal $\lambda \in M$, we have that $\mathrm{cf}(\lambda)^M = \mathrm{cf}(\lambda)^{M[G]}$. Prove that if $\mathbb{P}$ preserves cofinalities, then it preserves cardinals.

(40) If κ is a cardinal, we say that $\mathbb{P}$ has the κ-c.c. if every antichain in $\mathbb{P}$ has cardinality smaller than κ. (Thus, the c.c.c. is the $\aleph_1$-c.c.) If κ is a cardinal in M, we say that $\mathbb{P}$ *preserves cardinals* $\geq \kappa$ (or *preserves cardinals* $\leq \kappa$) if for every $\mathbb{P}$-generic filter G over M and every $\lambda \geq \kappa$ (or $\lambda \leq \kappa$), we have that $M \models$"λ is a cardinal" if and only if "$M[G] \models$"λ is a cardinal". Show that if $M \models$"κ is a regular cardinal" and $M \models$"$\mathbb{P}$ has the κ-c.c.", then $\mathbb{P}$ preserves cardinals $\geq \kappa$.

(41) What can you say about the chain condition of $\mathrm{Fn}(\omega, \omega_1^M)$, i.e., for which κ does this partial order have the κ-c.c.? Show that if G is $\mathrm{Fn}(\omega, \omega_1^M)$-generic over M, then $\aleph_1^{M[G]} = \aleph_2^M$.

(42) A partial order $\mathbb{P}$ is called λ-*closed* if whenever $\gamma < \lambda$ and $S := \{p_\xi\,;\, \xi < \gamma\}$ is a decreasing sequence of elements in $\mathbb{P}$, then there is a $q \in \mathbb{P}$ such that q is below all elements of S. Suppose that $\mathbb{P}$ is λ-closed, that $\alpha < \lambda$, β is any ordinal, that G is $\mathbb{P}$-generic over M, and that $f \in M[G]$ with $f : \alpha \to \beta$. Show that $f \in M$. Deduce that forcing with λ-closed forcing does not collapse any cardinals $\leq \lambda$.

(43) We write $\mathrm{Fn}(I, J, \lambda)$ for the partial order of partial functions p with $\mathrm{dom}(p) \subseteq I$, $\mathrm{ran}(p) \subseteq J$, and $M \models |\mathrm{dom}(p)| < \lambda$, ordered by reverse inclusion. Let $\mathbb{P} := \mathrm{Fn}(\aleph_\omega^M, 2, \aleph_\omega^M)$. Suppose that G is $\mathbb{P}$-generic over M. Show that in $M[G]$, the ordinal $\aleph_\omega^M$ is countable.