Topics in Set Theory
Lent Term 2018
Part III of the Mathematical Tripos
University of Cambridge
Prof. Dr. B. Löwe

Example Sheet #2

Example Classes.
#1. Thursday 1 February, 3–4pm, MR13.
#2. Thursday 15 February, 3–4pm, MR13.
#3. Thursday 1 March, 3–4pm, MR21.
#4. Wednesday 14 March, 3–4pm, MR21.
You hand in your work at the beginning of the Example Class.

(12) For a cardinal κ, let $\mathbf{H}_\kappa := \{x\,;\,|\mathrm{tcl}(x)| < \kappa\}$, the set of sets of hereditary size smaller than κ. Show that

  (a) $\mathbf{H}_{\aleph_1} \models \neg\mathsf{PowerSet}$,
  (b) $\mathbf{H}_{\aleph_1} \models \mathsf{Replacement}$.

(13) Suppose $M \subseteq N$ are transitive models of ZFC. Suppose that M and N disagree about the value of $\aleph_1$, i.e., there are ordinals $\alpha_M \neq \alpha_N$ such that $M \models$"$\alpha_M = \aleph_1$" and $N \models$"$\alpha_N = \aleph_1$". Show that there is some $x \in N\backslash M$ such that $N \models x \subseteq \mathbb{N}$.

(14) Suppose that φ is a formula with n free variables and A any set. Show that the set $\{s \in A^n\,;\,\varphi^A(s_1, ..., s_n)\}$ is in $\mathrm{Def}(A, n)$.

(15) Show the *Lévy Reflection Theorem*: if Φ is a finite set of formulas and α is any ordinal, then there is some $\beta > \alpha$ such that all formulas in Φ are absolute for $\mathbf{V}_\beta$.

(*Hint.* Assume, without loss of generality that Φ is closed under subformulas and show a Tarski-Vaught style criterion for absoluteness by induction on the formula complexity.)

(16) What properties of $\mathbf{V}_\beta$ did you use in (15)? Can you generalise the Reflection Theorem to obtain absoluteness for other structures than the $\mathbf{V}_\alpha$?

(17) Use the Reflection Theorem to show that if φ is any formula, x is any set and $\alpha_1, ..., \alpha_n$ are ordinals such that x is uniquely determined by φ with parameters $\alpha_1, ..., \alpha_n$, i.e.,

$$\forall z(z = x \leftrightarrow \varphi(z, \alpha_1, ..., \alpha_n)),$$

then $x \in \mathbf{OD}$.

(18) Show that for each α, $\mathbf{V}_\alpha$ and $\mathbf{V}_\alpha \cap \mathbf{OD}$ are in $\mathbf{OD}$.

(*Remark.* Remember that these were the crucial components of the proof that if $\mathbf{V} \neq \mathbf{OD}$, then $\mathbf{OD}$ is not a model of Extensionality: $\mathbf{V}_\alpha$ and $\mathbf{V}_\alpha \cap \mathbf{OD}$ are co-extensional in $\mathbf{OD}$.)

(19) Show that

   (a) $(\mathbf{HOD}, \in) \models \mathsf{Union}$;
   (b) $(\mathbf{HOD}, \in) \models \mathsf{Separation}$;
   (c) $(\mathbf{HOD}, \in) \models \mathsf{PowerSet}$.

(20) Show that

   (a) $(\mathbf{L}, \in) \models \mathsf{Union}$;
   (b) $(\mathbf{L}, \in) \models \mathsf{PowerSet}$.

(21) Show in ZFC that for $\alpha > \omega$, $|\mathbf{V}_\alpha| = |\mathbf{L}_\alpha|$ if and only if α is a beth fixed point.

(22) Show in $\mathsf{ZFC} + \mathbf{V}{=}\mathbf{L}$ that for $\alpha > \omega$, $\mathbf{V}_\alpha = \mathbf{L}_\alpha$ if and only if α is a beth fixed point.