Topics in Set Theory
Lent Term 2018
Part III of the Mathematical Tripos
University of Cambridge
Prof. Dr. B. Löwe

Example Sheet #1

Example Classes.
#1. Thursday 1 February, 3–4pm, MR21.
#2. Thursday 15 February, 3–4pm, MR13.
#3. Thursday 1 March, 3–4pm, MR21.
#4. Wednesday 14 March, 3–4pm, MR21.
You hand in your work at the beginning of the Example Class.

1. A cardinal κ is called an *aleph fixed point* if $\aleph_\kappa = \kappa$. Let μ be any regular cardinal. Show in ZFC that there is an aleph fixed point κ such that $\mathrm{cf}(\kappa) = \mu$.

2. The *von Neumann hierarchy* is defined by the following transfinite recursion:

$$\begin{aligned} \mathbf{V}_0 &:= \varnothing, \\ \mathbf{V}_{\alpha+1} &:= \wp(\mathbf{V}_\alpha), \text{ and} \\ \mathbf{V}_\lambda &:= \bigcup_{\alpha<\lambda} \mathbf{V}_\alpha \text{ (for limit ordinals } \lambda). \end{aligned}$$

   The *Mirimanoff rank* is defined by $\varrho(x) := \min\{\alpha\,;\, x \in \mathbf{V}_{\alpha+1}\}$. Show that for any ordinals $\alpha \leq \beta$, the following hold:

   (a) $\mathbf{V}_\alpha$ is transitive;
   (b) $\mathbf{V}_\alpha \subseteq \mathbf{V}_\beta$;
   (c) if $x \in y$, then $\varrho(x) < \varrho(y)$;
   (d) $\varrho(x) := \sup\{\varrho(y) + 1\,;\, y \in x\}$;
   (e) $\alpha = \{\gamma \in \mathbf{V}_\alpha\,;\, \gamma \text{ is an ordinal}\}$.

3. Suppose that $\lambda > \omega$ is a limit ordinal. Show that

   (a) $(\mathbf{V}_\lambda, \in) \models \mathsf{Union}$;
   (b) $(\mathbf{V}_\lambda, \in) \models \mathsf{Separation}$;
   (c) $(\mathbf{V}_\lambda, \in) \models \mathsf{PowerSet}$.

4. The representation theorem for wellorders says that every wellorder is isomorphic to a unique ordinal. This is a theorem proved in ZF, using the Axiom of Replacement. Give a concrete example of a wellorder $(X, R) \in \mathbf{V}_{\omega+\omega}$ that is not isomorphic to an ordinal $\alpha \in \mathbf{V}_{\omega+\omega}$.

5. Let γ be an ordinal such that $\mathbf{V}_\gamma \models \mathsf{ZFC}$. Show that γ is a cardinal. (We called cardinals like this *worldly cardinals*.)

   (*Note:* In order to violate Replacement, it is not enough to find $x \in \mathbf{V}_\gamma$ and a surjection from x onto γ. The surjection has to be definable!)

6. Let λ be a limit ordinal. We say that $C \subseteq \lambda$ is *closed* if for every α, if $C \cap \alpha$ is cofinal in α, then $\alpha \in C$. We say that C is *closed unbounded* or *club* in λ if it is closed and cofinal in λ.

   Suppose that κ is an inaccessible cardinal and show that the set $\{\lambda < \kappa\,;\, \lambda \text{ is a worldly cardinal}\}$ is club in κ.

7. We called a formula Δ_0 if it is in the closure of the quantifier-free formulas under the operations $\varphi \mapsto \neg\varphi$, $(\varphi, \psi) \mapsto \varphi \wedge \psi$, $(\varphi, \psi) \mapsto \varphi \vee \psi$, $(\varphi, \psi) \mapsto \varphi \rightarrow \psi$, and $\varphi \mapsto \exists x(x \in y \wedge \varphi)$. Check whether the following formulas are Δ_0 and give an argument for your answer:

   (a) $\exists x(\forall z(\neg z \in x) \wedge x \in y)$;
   (b) $(x = y) \vee (z \in x)$;
   (c) $\forall x(x \in y \rightarrow x \in z)$;
   (d) $\exists x(x \in y \wedge \neg x \in z)$;
   (e) $\exists x(x \in y \wedge \neg(\exists z(z \in y \wedge (z \in x \vee y \in x))))$.

8. Let T be any $\mathcal{L}_\in$-theory. We called a formula Δ_0^T if the theory T proves that it is equivalent to a Δ_0 formula. Show that the following concepts can be expressed by Δ_0^T-formulas for a reasonable choice of T; also, indicate what T you choose and why.

   (a) $z = \{x, y\}$;
   (b) $z = (x, y)$;
   (c) $z = y \times y$;
   (d) z is a function;
   (e) z is a group;
   (f) z is a linear order;
   (g) z is a set with exactly two elements.

9. Let $\Phi(x)$ be the formula expressing "x is an inaccessible cardinal" and let λ be a limit ordinal. Show that Φ is absolute for $\mathbf{V}_\lambda$.

10. We write $\mathsf{2IC}$ for the statement "there are $\kappa < \lambda$ that are both inaccessible". Show that the theory $\mathsf{ZFC} + \mathsf{IC}$ does not prove $\mathsf{2IC}$.

11. Work in $\mathsf{ZFC} + \mathsf{IC}$ and show that there is no formula Φ such that

    (a) $\Phi(x)$ holds if and only if x is a worldly cardinal and
    (b) Φ is absolute for transitive models of ZFC.

    (*Hint.* Construct a transitive model that cannot contain any worldly cardinals, but some of its ordinals satisfy Φ.)