Classification in Descriptive Set Theory

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Outline

Polish Spaces

A Polish space is a separable metrizable space
which has a complete metric.

Polish spaces are widespread in mathematics.
Many mathematical objects can be viewed as elements of some Polish space.

The following are Polish spaces:
$2 = \{0, 1\}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}$, $I = [0, 1]$, the circle $\mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \}$,
any compact metrizable space, any separable Banach space.

A countable product of Polish spaces is a Polish space,
and so $\mathbb{R}^{\mathbb{N}}$, $\mathbb{C}^{\mathbb{N}}$, $I^{\mathbb{N}}$ (the Hilbert cube),
$2^{\mathbb{N}}$ (the Cantor space, homeomorphic to the Cantor $1/3$-set),
$\mathbb{N}^{\mathbb{N}}$ (the Baire space, homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$) are Polish spaces.

More Polish spaces

If X is a compact metric space and Y is a Polish space, the space $C(X,Y)$ of continuous functions from X into Y with the uniform metric is Polish.

If X is Polish and $A \subseteq X$ is $\mathbf{G}_\delta$ (in particular open or closed), A is Polish in the induced topology (with a different metric if A is not closed).

The power set of $\mathbb{N}^{<\mathbb{N}}$ (finite sequences of natural numbers) is homeomorphic to $2^{\mathbb{N}}$.
$\mathbf{Tr} = \{\, T \subseteq \mathbb{N}^{<\mathbb{N}} \mid \forall s,t\ (s \in T \mathbin{\&} t \sqsubset s \implies t \in T)\,\}$ is closed in it, hence Polish.
This is the set of (descriptive set-theoretic) trees.

Borel sets

If X is any topological space the Borel subsets of X
are the members of the smallest σ-algebra containing the open sets.

Hence the Borel sets are generated by the open and closed sets
by closing under countable unions and intersections.

$$\mathbf{G} = \{\text{open sets}\} = \mathbf{\Sigma}^0_1, \qquad \mathbf{F} = \{\text{closed sets}\} = \mathbf{\Pi}^0_1$$
$$\mathbf{F}_\sigma = \{\text{countable unions of closed sets}\} = \mathbf{\Sigma}^0_2,$$
$$\mathbf{G}_\delta = \{\text{countable intersections of open sets}\} = \mathbf{\Pi}^0_2,$$
$$\mathbf{G}_{\delta\sigma} = \{\text{countable unions of } \mathbf{\Pi}^0_2 \text{ sets}\} = \mathbf{\Sigma}^0_3,$$
$$\mathbf{F}_{\sigma\delta} = \{\text{countable intersections of } \mathbf{\Sigma}^0_2 \text{ sets}\} = \mathbf{\Pi}^0_3,$$
$$\dots\dots\dots\dots\dots\dots$$

The Borel hierarchy

$$\mathbf{\Sigma}^0_1 = \{\text{open sets}\}, \qquad \mathbf{\Pi}^0_1 = \{\text{closed sets}\}$$
$$\mathbf{\Sigma}^0_\alpha = \{\text{countable unions of } \mathbf{\Pi}^0_\beta \text{ sets with } \beta < \alpha\},$$
$$\mathbf{\Pi}^0_\alpha = \{\text{countable intersections of } \mathbf{\Sigma}^0_\beta \text{ sets with } \beta < \alpha\},$$
$$\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha.$$

$$\begin{array}{ccccccccc} & \mathbf{\Sigma}^0_1 & & \mathbf{\Sigma}^0_2 & & & \mathbf{\Sigma}^0_\alpha & & \\ \mathbf{\Delta}^0_1 & & \mathbf{\Delta}^0_2 & & \dots & \mathbf{\Delta}^0_\alpha & & \mathbf{\Delta}^0_{\alpha+1} & \dots \\ & \mathbf{\Pi}^0_1 & & \mathbf{\Pi}^0_2 & & & \mathbf{\Pi}^0_\alpha & & \end{array}$$

$$\bigcup_{1\le\alpha<\aleph_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{1\le\alpha<\aleph_1} \mathbf{\Pi}^0_\alpha = \bigcup_{1\le\alpha<\aleph_1} \mathbf{\Delta}^0_\alpha = \{\text{Borel sets}\}.$$

The projective hierarchy

The continuous image of a Borel set is not in general Borel.

$$\begin{gathered}
\boldsymbol{\Sigma}^1_1 = \{\text{continuous images of Borel sets}\} = \{\text{analytic sets}\},\\
\boldsymbol{\Pi}^1_n = \{\text{complements of } \boldsymbol{\Sigma}^1_n \text{ sets}\},\\
\boldsymbol{\Sigma}^1_{n+1} = \{\text{continuous images of } \boldsymbol{\Pi}^1_n \text{ sets}\},\\
\boldsymbol{\Delta}^1_n = \boldsymbol{\Sigma}^1_n \cap \boldsymbol{\Pi}^1_n.
\end{gathered}$$

$$\begin{array}{ccccccccc}
 & \boldsymbol{\Sigma}^1_1 = \mathbf{A} & & \boldsymbol{\Sigma}^1_2 = \mathbf{PCA} & & & \boldsymbol{\Sigma}^1_n & & \\
\boldsymbol{\Delta}^1_1 & & \boldsymbol{\Delta}^1_2 & & \cdots & \boldsymbol{\Delta}^1_n & & \boldsymbol{\Delta}^1_{n+1} & \cdots \\
 & \boldsymbol{\Pi}^1_1 = \mathbf{CA} & & \boldsymbol{\Pi}^1_2 = \mathbf{CPCA} & & & \boldsymbol{\Pi}^1_n & &
\end{array}$$

Souslin's Theorem: in Polish spaces $\boldsymbol{\Delta}^1_1 = \{\text{Borel sets}\}$.

Classifying natural sets

- What is a "natural" set?
  A set arising in mathematical practice.
- What does it mean to classify a set?
  To find the lowest level in these hierarchies where it appears.

Classifying natural sets gives precise mathematical explanations to empirical phenomena such as:

- checking whether an object belongs to A is "more difficult" than checking whether it belongs to B;
- we cannot find a "simple" definition of A.

We want to pin down the complexity of a set,
e.g. proving that it is true $\mathbf{\Pi}^0_3$, i.e. $\mathbf{\Pi}^0_3$ but not $\mathbf{\Sigma}^0_3$.

Wadge reducibility

$A \subseteq X$, $B \subseteq Y$ (X, Y Polish)

A is Wadge-reducible to B ($A \leq_{\mathrm{W}} B$)
iff $\exists f : X \to Y$ continuous, $A = f^{-1}(B)$, i.e.

$$x \in A \iff f(x) \in B \quad \text{for each } x \in X.$$

We continuously reduce membership in A to membership in B.

B is at least as complicated as A.

$\leq_{\mathrm{W}}$ is reflexive and transitive.

Completeness

Let $\mathbf{\Gamma}$ be one of the classes of the Borel or the projective hierarchy.
If $B \in \mathbf{\Gamma}$ and $A \leq_{\mathrm{W}} B$, then $A \in \mathbf{\Gamma}$.

Let X be a Polish space. $B \subseteq X$ is $\mathbf{\Gamma}$-hard iff
$\forall A \subseteq Y$, Y (zero-dimensional) Polish space, $A \in \mathbf{\Gamma}$, $A \leq_{\mathrm{W}} B$.

B is $\mathbf{\Gamma}$-complete if it belongs to $\mathbf{\Gamma}$ and is $\mathbf{\Gamma}$-hard.

A $\mathbf{\Gamma}$-complete set is true $\mathbf{\Gamma}$ (for Borel classes the reversal holds).

Completeness in practice

Suppose we are given $A \subseteq X$, with X Polish,
and we want to establish that it is $\mathbf{\Pi}^1_1$-complete.
We need to show:

1. A is $\mathbf{\Pi}^1_1$;
2. A is $\mathbf{\Pi}^1_1$-hard.

1. establishes the upper bound.
This is usually (but not always) the easiest step.
Often the standard definition of A shows that it is $\mathbf{\Pi}^1_1$.

2. establishes the lower bound.
The definition of $\mathbf{\Pi}^1_1$-hardness calls for considering all $\mathbf{\Pi}^1_1$ sets.
In practice this is not needed: it suffices to show that $P \leq_{\mathrm{W}} A$
for some set P which is already known to be $\mathbf{\Pi}^1_1$-complete.

Some basic examples

In practice, a member of a fairly small collection of (mostly combinatorial) examples is used to prove hardness.

$$\begin{aligned} Q_2 &= \{\, \alpha \in 2^{\mathbb{N}} \mid \forall^\infty n\ \alpha(n) = 0 \,\} && \text{is } \mathbf{\Sigma}^0_2\text{-complete;} \\ P_3 &= \{\, \alpha \in 2^{\mathbb{N}\times\mathbb{N}} \mid \forall m\ \forall^\infty n\ \alpha(m,n) = 0 \,\} && \text{is } \mathbf{\Pi}^0_3\text{-complete;} \\ S_3^* &= \{\, \alpha \in 2^{\mathbb{N}\times\mathbb{N}} \mid \forall^\infty m\ \exists n\ \alpha(m,n) = 0 \,\} && \text{is } \mathbf{\Sigma}^0_3\text{-complete;} \\ \mathbf{WF} &= \{\, T \in \mathbf{Tr} \mid T \text{ has no infinite paths} \,\} && \text{is } \mathbf{\Pi}^1_1\text{-complete.} \end{aligned}$$

$$\alpha \in Q_2 \iff \exists M \in \mathbb{N}\, \forall n > M\ \alpha(n) = 0$$

$$T \notin \mathbf{WF} \iff \underbrace{\exists \alpha \in \mathbb{N}^{\mathbb{N}}\ \underbrace{\forall n \in \mathbb{N}\ \alpha \restriction n \in T}_{\mathbf{\Pi}^0_1}}_{\mathbf{\Sigma}^1_1}$$

Examples I: sequences

- $\ell^p = \{ (x_k) \in I^{\mathbb{N}} \mid \sum_{k=0}^{\infty} {x_k}^p < +\infty \}$ is $\mathbf{\Sigma}^0_2$-complete.
- $c_0 = \{ (x_k) \in I^{\mathbb{N}} \mid \lim_{k\to\infty} x_k = 0 \}$ is $\mathbf{\Pi}^0_3$-complete.

c_0 is $\mathbf{\Pi}^0_3$

$$c_0 = \underbrace{\bigcap_{\varepsilon\in\mathbb{Q}^+} \underbrace{\bigcup_{N\in\mathbb{N}} \underbrace{\bigcap_{k\geq N} \left\{ (x_k) \in I^{\mathbb{N}} \mid x_k \leq \varepsilon \right\}}_{\mathbf{\Pi}^0_1}}_{\mathbf{\Sigma}^0_2}}_{\mathbf{\Pi}^0_3}.$$

c_0 is Π^0_3-hard

We show $P_3 \leq_{\mathrm{W}} c_0$ where $P_3 = \{\, \alpha \in 2^{\mathbb{N}\times\mathbb{N}} \mid \forall m\, \forall^\infty n\, \alpha(m,n) = 0 \,\}$.

We need $f : 2^{\mathbb{N}\times\mathbb{N}} \to I^{\mathbb{N}}$ continuous such that $\alpha \in P_3 \iff f(\alpha) \in c_0$.

Let $f(\alpha) = (x_{(m,k)})$ with $x_{(m,k)} = 2^{k-m} \sum_{n \geq k} \frac{\alpha(m,n)}{2^{n+1}} \leq 2^{-m}$.

If $\alpha \in P_3$ then fix $\varepsilon \in \mathbb{Q}^+$ and M such that $2^{-M} \leq \varepsilon$.
Let N be such that $\forall m < M\, \forall n \geq N\, \alpha(m,n) = 0$.

- if $m < M$ and $k \geq N$ then $x_{(m,k)} = 0 < \varepsilon$;
- if $m \geq M$ then $x_{(m,k)} \leq 2^{-m} \leq 2^{-M} \leq \varepsilon$ for all k.

Thus $x_{(m,k)} > \varepsilon$ only for finitely many pairs (m,k) and $f(\alpha) \in c_0$.

If $\alpha \notin P_3$ let m be such that $\exists^\infty n\, \alpha(m,n) = 1$.
For every such n we have $x_{(m,n)} \geq 2^{n-m} \frac{1}{2^{n+1}} = 2^{-m-1}$.
Thus $f(\alpha) \notin c_0$.

Examples II: functions

- $C^n(I)$ and $C^\infty(I)$ are $\mathbf{\Pi}^0_3$-complete in $C(I,\mathbb{R})$.
- $\{ f \in C(I,\mathbb{R}) \mid f$ is analytic $\}$ is $\mathbf{\Sigma}^0_2$-complete.
- (Mazurkiewicz 1936) $\{ f \in C(I,\mathbb{R}) \mid \forall x \in I\ f'(x)$ exists $\}$ is $\mathbf{\Pi}^1_1$-complete.
- (Mauldin 1979) $\{ f \in C(I,\mathbb{R}) \mid \forall x \in I\ f'(x)$ does not exist $\}$ is $\mathbf{\Pi}^1_1$-complete.
- (Kechris 1984) $\{ f \in C(I,\mathbb{R}) \mid \forall y \in \mathbb{R}\ \exists x \in I\ f'(x) = y \}$ is $\mathbf{\Pi}^1_2$-complete.
- (Woodin 1990) If $f \in C(I,\mathbb{R})$ say that f satisfies Rolle's Theorem if for all $a, b \in I$ with $a < b$ and $f(a) = f(b)$ there exists $c \in (a,b)$ such that $f'(c) = 0$. The set of such f's is $\mathbf{\Sigma}^1_1$-complete.
- (Woodin 1990) If $f \in C(I,\mathbb{R})$ say that f satisfies the Mean Value Theorem if for all $a, b \in I$ with $a < b$ there exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. The set of such f's is $\mathbf{\Pi}^1_2$-complete.

Binary relations in Polish spaces

A binary relation on a Polish space X is a subset of $X \times X$.

We can find its position in the descriptive set theoretic hierarchies.

This approach does not take into account
the particular features of a binary relation.

The next definition has been introduced and studied in depth
in the context of equivalence relations:
it is the starting point of the rich subject of
"Borel reducibility for equivalence relations"
aka "invariant descriptive set theory".

Borel reducibility

If E and F are binary relations on Polish spaces X and Y,
a reduction of E to F is a function $f : X \to Y$ such that

$$\forall x_0, x_1 \in X (x_0 \, E \, x_1 \iff f(x_0) \, F \, f(x_1)).$$

If f is Borel we say that E is Borel reducible to F: $E \leq_{\mathrm{B}} F$

E, F equivalence relations on X, Y. What does it mean that $E \leq_{\mathrm{B}} F$?

- we can assign in a Borel way F-equivalence classes
  as complete invariants for the e.r. E;
- E has a simpler classification problem than F:
  invariants for F work for E (composing with the reduction);
- $\exists \Phi : X/E \xrightarrow{1-1} Y/F$ that can be lifted to a Borel map from X to Y:
  the effective cardinality of the quotient space X/E is less than
  or equal to the effective cardinality of the quotient space Y/F.

Smooth equivalence relations

An e.r. on a Polish space is smooth (or concretely classifiable, or tame) if it is Borel reducible to equality on some Polish space.

A smooth e.r. is considered to be very simple, since it admits "concrete" objects as complete invariants.

A smooth e.r. is Borel.

An example of a smooth e.r. is similarity between $n \times n$ complex matrices: assign to each matrix its Jordan normal form.

S_∞-universal equivalence relations

$\mathcal{L}$ a countable relational language:
$X_\mathcal{L}$ the Polish space of (codes for) $\mathcal{L}$-structures with universe $\mathbb{N}$
(it is homeomorphic to $2^\mathbb{N}$).

$\cong_\mathcal{L}$ isomorphism on $X_\mathcal{L}$.

- an e.r. E on a Polish space is classifiable by countable structures if $E \leq_\mathrm{B} \cong_\mathcal{L}$ for some $\mathcal{L}$;
- E is S_∞-universal if moreover $\cong_\mathcal{L} \leq_\mathrm{B} E$ for every $\mathcal{L}$.

An e.r. is S_∞-universal iff it is as complicated as any orbit e.r.
of a continuous action of the infinite symmetric group S_∞ can be.

A S_∞-universal e.r. is $\mathbf{\Sigma}^1_1$ and not Borel,
yet its equivalence classes are Borel.

Isomorphism between trees is S_∞-universal (H.Friedman-Stanley 1989).
Homeomorphism between compact subsets of $2^\mathbb{N}$ is S_∞-universal
(Camerlo-Gao 2001).

Σ^1_1-complete equivalence relations

An e.r. E on a Polish space is $\boldsymbol{\Sigma}^1_1$-complete if it is $\boldsymbol{\Sigma}^1_1$ and $F \leq_{\mathrm{B}} E$ for any $\boldsymbol{\Sigma}^1_1$ e.r. F on a Polish space.

A $\boldsymbol{\Sigma}^1_1$-complete e.r. is immensely more complicated than any e.r. induced by any Polish group action (e.g. uncountably many of its equivalence classes are not Borel).

Biembeddability between countable partial orders is $\boldsymbol{\Sigma}^1_1$-complete (Louveau-Rosendal 2005).

For any $\boldsymbol{\Sigma}^1_1$ e.r. F on a Polish space there is a Borel class $\mathcal{B}$ of countable partial orders invariant under isomorphism such that F is Borel bireducible with biembeddability restricted to $\mathcal{B}$ (S.Friedman-Motto Ros 2011).

Countable linear orders

Isomorphism between countable countable linear orders is S_∞-universal (H.Friedman-Stanley 1989).

Biembeddability between countable linear orders
has $\aleph_1$ equivalence classes and hence is not $\mathbf{\Sigma}^1_1$-complete.

A colored linear order is a linear order $(L, \leq_L)$ with a coloring $c : L \to \mathbb{N}$.
Embeddings between colored linear orders preserve order and color.

Biembeddability between countable colored linear orders is $\mathbf{\Sigma}^1_1$-complete (M-Rosendal 2004, Camerlo 2005).

For any $\mathbf{\Sigma}^1_1$ e.r. F on a Polish space there is a Borel class $\mathcal{B}$
of countable colored linear orders invariant under isomorphism
such that F is Borel bireducible with biembeddability restricted to $\mathcal{B}$
(Camerlo-M-Motto Ros, 201?).

The end

Thank you for your attention!