Tutorial 1.3: Combinatorial Set Theory

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I. Generalizing Ramsey's Theorem

Our proof of Ramsey's Theorem for pairs was modeled on a proof by Erdős, and its extension to the Infinite Ramsey Theorem on the Stepping Up Lemma of Erdős and Rado.

Theorem 1

For all infinite κ, $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$.

Theorem 2 (The Stepping Up Lemma of Erdős-Rado 1956 [9])

Assume $\kappa \geq \omega$, $1 \leq r < \omega$, $\gamma < \kappa$ and $\kappa \to (\alpha_\xi)^r_\gamma$. Then

$$\left(2^{<\kappa}\right)^+ \to (\alpha_\xi + 1)^{r+1}_\gamma.$$

I. Generalizing Ramsey's Theorem

The cardinal $\aleph_\omega$ is the least ordinal bigger than $\aleph_n$ for all $n < \omega$. Moreover $\aleph_\omega = \bigcup_{n<\omega} \aleph_n$. The *cofinality* of a cardinal κ, denoted $\mathrm{cf}(\kappa)$ is the smallest ordinal λ such that κ is the union of λ many smaller sets. So $\mathrm{cf}(\aleph_\omega) = \omega$.

Theorem 3 (Erdős-Rado 1956 [9])

For all infinite κ and all $\gamma < \mathrm{cf}(\kappa)$,

$$\exp_{r-2}\left(2^{<\kappa}\right)^{+} \longrightarrow (\kappa + (r-1))^{r}_{\gamma}.$$

I. Generalizing Ramsey's Theorem

Erdős talked his partition problems at the first post-forcing set theory meeting (UCLA 1967), and he wrote up problems with Hajnal that only appeared in 1971 [7].

Question (Erdős-Hajnal 1971: Problem 10)

Assume the GCH. Does

$$\omega_{\xi+1} \longrightarrow (\alpha, \alpha)^2_2$$

hold for $\alpha < \omega_{\xi+1}$?

I. Generalizing Ramsey's Theorem

Theorem 4 (Baumgartner-Hajnal 1973 [2])

For all $\alpha < \omega_1$ and $m < \omega$,

$$\omega_1 \to (\alpha)^2_m.$$

Todorcevic [16] generalized the theorem in 1985 to partial orders; the critical case is the positive result for non-special trees.

Theorem 5 (Baumgartner-Hajnal-Todorcevic 1993 [3])

For all regular uncountable κ, $m < \omega$ and $\rho < \omega_1$ with $2^{|\rho|} < \kappa$,

$$(2^{<\kappa})^+ \to (\kappa + \rho)^2_m.$$

I. Generalizing Ramsey's Theorem

Shelah was able to increase the number of colors in the palette by looking at larger cardinals.

Theorem 6 (Shelah 2003 [15])

If λ is a strongly compact cardinal, $\zeta, \mu < \lambda$, and κ is a regular cardinal $> \lambda$, then

$$(2^{<\kappa})^+ \to (\kappa + \zeta)^2_\mu.$$

I. Generalizing Ramsey's Theorem

Question (Foreman-Hajnal 2003: Problem 1)

Is $\kappa^+ \longrightarrow (\kappa \cdot 2)^2_\omega$ consistently true for any κ?

Foreman and Hajnal [10] have proved $\kappa^+ \longrightarrow (\rho)^2_m$ for finite m and ρ below a large bound.

II. Uncountable partition ordinals

Theorem 7 (Hajnal 1960 [11])

If CH holds, then ${\omega_1}^2$ and $\omega_1 \cdot \omega$ are not partition ordinals:

$$\mathrm{CH} \vdash {\omega_1}^2 \not\rightarrow ({\omega_1}^2, 3)^2 \text{ and } \omega_1 \cdot \omega \not\rightarrow (\omega_1 \cdot \omega, 3)^2.$$

Theorem 8 (Baumgartner 1989 [1])

If $\mathrm{MA}_{\aleph_1}$ holds, then $\omega_1 \cdot \omega$ and $\omega_1 \cdot \omega^2$ are partition ordinals:

$$\mathrm{MA}_{\aleph_1} \vdash \omega_1 \cdot \omega \longrightarrow (\omega_1 \cdot \omega, 3)^2 \text{ and } \omega_1 \cdot \omega^2 \longrightarrow (\omega_1 \cdot \omega^2, 3)^2$$

Question

Is it consistent that ${\omega_1}^2 \longrightarrow ({\omega_1}^2, 3)^2$?

III. Partitions of triples

Ordinal partition relations have been defined for partitions of r-uniform hypergraphs, e.g. $\alpha \rightarrow (\beta, \gamma)^r$ holds if for every ordered set V of order type α and every $c : [V]^3 \rightarrow 2$, there is either

a subset $X \subseteq V$ order-isomorphic to β for which c is constantly 0 on $[X]^r$ (*homogenous for color* 0) ; or

a subset Y with Y order isomorphic to γ for which c is constantly 1 on $[Y]^r$ (homogeneous for color 1).

Theorem 9 (Jones 2007 [12])

For all $m, n < \omega$,

$$\omega_1 \rightarrow (\omega + m, n)^3.$$

III. Partitions of triples

Question

Does $\omega_1 \to (\alpha, n)^3$ for all $\alpha < \omega_1$?

In unpublished work, Jones has shown that $\omega_1 \to (\omega + \omega + 1, n)^3$, so the simplest open instance of the question is

$$\omega_1 \to (\omega + \omega + 2, 4)^3.$$

Jones [13] also has generalizations to partial orders.

IV. Square bracket partitions

Square bracket partition relations were introduced to express strong negations of ordinary partition relations by providing a way to indicate that there is a coloring so that no suitably large set omits any color.

IV. Square bracket partitions

Definition

The square bracket partition $\kappa \to [\alpha_\nu]^r_\mu$ if for every coloring $c : [\kappa]^r \to \mu$, there is some $\nu < \mu$ and a subset $X \subseteq \kappa$ of order type α_ν so that c omits color on $[X]^r$.

Theorem 10 (Erdős-Hajnal-Rado 1965 [8])

If $2^\kappa = \kappa^+$, then $\kappa^+ \not\to [\kappa^+]^2_{\kappa^+}$.

IV. Square bracket partitions

Question (Todorcevic)

If κ is an infinite cardinal, does

$$\kappa^+ \nrightarrow [\kappa^+]^2_{\kappa^+}?$$

Theorem 11 (Todorcevic 1987 [17])

$$\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}.$$

The paper in which Todorcevic proved this theorem introduced his method of walks on ordinals, and includes his construction of a Suslin tree from a Cohen real (the existence was first proved by Shelah).

IV. Square bracket partitions

Theorem 12 (Todorcevic 1987 [17])

For uncountable κ, if κ^+ has a non-reflecting stationary set, then

$$\kappa^+ \nrightarrow [\kappa^+]^2_{\kappa^+}.$$

A set $A \subseteq \lambda$ is *stationary in* λ if it intersects every set which is unbounded in λ and closed under taking limits. For $\gamma < \lambda$ of uncountable cofinality, *A reflects at* γ if its intersection with γ is stationary in γ. It is *non-reflecting* if it fails to reflect for all smaller γ of uncountable cofinality.

IV. Square bracket partitions

Theorem 13 (Todorcevic 1987 [17], see Burke-Magidor [4])

It follows from a stepping up argument and pcf theory that

$$\aleph_{\omega+1} \nrightarrow [\aleph_{\omega+1}]^2_{\aleph_{\omega+1}}.$$

Theorem 14 (Eisworth 2010 [5])

If μ is singular and $\mu^+ \rightarrow [\mu^+]^2_{\mu^+}$, then there is a regular cardinal $\theta < \mu$ such that any fewer than $\mathrm{cf}(\mu)$ many stationary subsets of $S^{\mu^+}_{\geq\theta} = \{\gamma < \mu^+ : \mathrm{cf}(\gamma) \geq \theta\}$ must reflect simultaneously.

V. Products of trees

The Halpern-Läuchli Theorem for partitions of level sets of a finite product of finitely branch trees which was motivated by a problem on weak forms of the Axiom of Choice.

The original statement of the Halpern-Läuchli Theorem requires more notation than I want to introduce. I recommend *Ramsey spaces* by Todorcevic [18] to learn about it.

V. Products of trees

Notation

Let $\prod^A \vec{T}$ be an abbreviation for $\bigcup_{n\in A} \prod_{i<\omega} T_i(n)$.

Theorem 15 (Laver's HL_ω Theorem 1984 [14])

If $\vec{T} = \langle T_i : i < \omega \rangle$ is a sequence of rooted finitely branching perfect trees of height ω and $\prod^\omega \vec{T} = G_0 \cup G_1$, then there are $\delta < 2$, an infinite subset $A \subseteq \omega$, and downwards closed perfect subtrees T_i' of T_i for $i < \omega$ with $\prod^A T_i' \subseteq G_\delta$.

V. Products of trees

Definition

Let $d \leq \omega$ be given and fix a sequence $\vec{T} = \langle T_i : i < d \rangle$ of perfect subtrees of ${}^{\omega>}2$, and when $d = \omega$, assume $\lim_{i<\omega} |\text{stem } T_i| = \omega$.

For $n < \omega$, call $\vec{X} = \langle X_i \subseteq T_i : i < d \rangle$ an *n-dense sequence* if all the points in $\bigcup X_i$ are on the same level $m > n$ and for all $y \in T_i(n)$ there is an $x \in X_i$ with $y < x$.

Definition

If $t \in T$, then T_t is the subtree of nodes comparable with t.

V. Products of trees

Question (Laver's Conjecture)

Does the strong version of the infinite Halpern Lauchli Theorem hold? Specifically if $\bigcup_{n<\omega} \prod \langle T_i(n) : i < \omega \rangle = G_0 \cup G_1$, then is it necessarily the case that either

(i) there is, for each n, an n-dense D such that $\prod D \subset G_0$; or

(ii) there is a $\langle t_i \in T_i : i < \omega \rangle$ such that (i) holds for $\langle (T_i)_{t_i} : i < \omega \rangle$ for G_1?

VI. References

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