Tutorial 1.2: Combinatorial Set Theory

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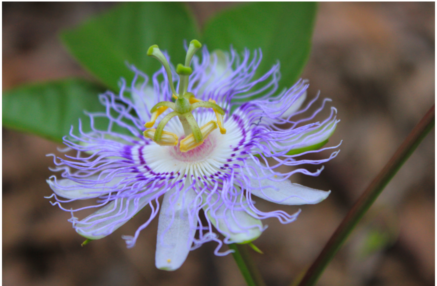
ESSLLI in Ljubljana, Slovenia, August 2, 2011



I. Overview

- A graph coloring problem for countable ordinals
- An ordinal where every coloring admits either large edgeless set or a triangle
- An ordinal with a coloring where neither exist
- Related results and open questions

II. A graph coloring problem

Definition

The *ordinal partition relation* $\alpha \to (\beta, \gamma)^2$ holds if and only if for every set V order isomorphic to α and every coloring $c : [V]^2 \to 2$, there is either

> a subset $X \subseteq V$ order-isomorphic to β for which c is constantly 0 on $[X]^2$ (*homogenous for color* 0) ; or
>
> a subset Y with Y order isomorphic to γ for which c is constantly 1 on $[Y]^2$ (homogeneous for color 1).

Note (V, E) is a graph for $E = \{\{v_0, v_1\} \in V \mid c(\{v_0, v_1\}) = 1\}$.

II. A graph coloring problem

Lemma 1

If $\alpha > \omega$ is a countably infinite ordinal, then $\alpha \not\to (\omega + 1, \omega)^2$.

Proof.

Let $<$ be the usual order on α and let $<'$ be a well-ordering of α in type ω.
Define $c : [V] \to 2$ by $c(\{v_0, v_1\}) = 0$ if and only $<$ and $<'$ agree on $\{v_0, v_1\}$.
If c is constantly 0 on $[X]^2$, then X has type at most ω.
If c is constantly 1 on $[Y]^2$, then Y is finite since the ordinals are well-founded. $\square$

III. A positive example

On a trip to Israel in 1956, Erdős visited Ernst Specker and talked about ordinal partition relations. Before Erdős continued on his way to Israel, Specker had proved a positive partition relation.

Theorem 2 (Specker 1957 [7])

$\omega^2 \rightarrow (\omega^2, m)^2$ for all $m < \omega$.

IV. Overview of the proof

The modern proof (Galvin, Hajnal, Haddad + Sabbagh: see [8])

- represents the order type ω^2 by $[\omega]^2_<$;
- identifies unavoidable patterns;
- canonizes the "same color" equivalence relations obtained from the unavoidable patterns
- shows there is a collection built from H of order type ω^2 using only unavoidable patterns (gives outcome for the infinite color).
- shows for any unavoidable pattern there are arbitarily large finite sets so each pair has the pattern using elements from H (gives outcome for the finite color);

V. Representation, unavoidable patterns

The set $[\omega]^2$ regarded as increasing sequences has order type $\omega^2 = \omega \cdot \omega$ under the lexicographic order $\leq_{\text{lex}}$.

Let us draw an arrow, $\nearrow$, to represent one copy of ω.

Then $\omega + \omega = \omega \cdot 2$ is $\nearrow \quad \nearrow$.

So $\omega \cdot 3$ is $\nearrow \quad \nearrow \quad \nearrow$,

$\omega \cdot n$ is $\nearrow \quad \nearrow \quad \nearrow \quad \ldots \quad \nearrow$, and

$\omega^2 = \omega \cdot \omega$ is $\nearrow \quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \quad \ldots$.

We identify an unordered pair $\{m, n\}$ with $m < n$ with the increasing sequence $\langle m, n \rangle$. Then we can think of m telling us which copy of ω we are in and n telling us at which point in the copy we are located.

V. Representation, unavoidable patterns

Fix a coloring $c : \left[[\omega]^2\right]^2 \to 2$. Define auxillary functions:
$C_0 : [\omega]^3 \to 2$ is defined by

$$C_0(\{i,j,k\}_<) = c(\{i,j\},\{i,k\}). \text{ (agree)}$$

$C_1 : [\omega]^4 \to 2$ is defined by

$$C_1(\{i_0,i_1,i_2,i_3\}_<) = c(\{i_0,i_1\},\{i_2,i_3\}). \quad (\triangle\ \triangle\ \square\square \text{ increase})$$

$C_2 : [\omega]^4 \to 2$ is defined by

$$C_2(\{i_0,i_1,i_2,i_3\}_<) = c(\{i_0,i_2\},\{i_1,i_3\}). \quad (\triangle\square\ \triangle\ \square \text{ alternate})$$

$C_3 : [\omega]^4 \to 2$ is defined by

$$C_3(\{i_0,i_1,i_2,i_3\}_<) = c(\{i_0,i_3\},\{i_1,i_2\}). \quad (\triangle\square\square\ \triangle \text{ enclose})$$

VI. Canonization for unavoidable patterns

Apply Ramsey's Theorem to each of C_0, C_1, C_2, C_3 in turn to get an infinite set $H \subseteq \omega$ and $\delta_0, \delta_1, \delta_2, \delta_3$ so that C_i is constantly δ_i on $[H]^3$ (or $[H]^4$).

Thus if $u = \{u_0, u_1\} \subseteq H$ and $v = \{v_0, v_1\} \subseteq H$ form an unavoidable pair, then

$c(\{u, v\}) = \delta_0$ if $u_0 = v_0$ (agree);

$c(\{u, v\}) = \delta_1$ if the pair in increasing;

$c(\{u, v\}) = \delta_2$ if it alternates; and

$c(\{u, v\}) = \delta_3$ if one encloses the other.

VII. A large set from H with unavoidable pairs

Partition $H = \bigcup H_j$ into disjoint infinite pieces.

Let $X = \{\{u_0, u_1\}_< \in [H]^2 \mid u_0 \in H_0 \wedge u_1 \in H_{u_0}\}$.

If $\{u_0, u_1\} \in X$ and $\{v_0, v_1\} \in X$ have non-empty intersection, then $u_0 = u_1$ (agree).

If $\{u_0, u_1\} \in X$ and $\{v_0, v_1\} \in X$ are disjoint, then either they increase, alternate or enclose.

Thus all pairs from X are unavoidable, so if $\delta_0 = \delta_1 = \delta_2 = \delta_3 = 0$, then X is a subset of order type ω^2 homogeneous in color 0.

VIII. Trios for the unavoidable patterns

Suppose $h_0 < x_1 < h_2 < h_3 < h_4 < h_5$ are from H.

(agree) $\{h_0, h_1\}$, $\{h_0, h_2\}$, $\{h_0, h_3\}$

(increase) $$\begin{array}{cccccc} \{h_0, & h_1\} & & & & \\ & & \{h_2, & h_3\} & & \\ & & & & \{h_4, & h_5\} \end{array}$$

(alternate) $$\begin{array}{cccccc} \{h_0, & & & h_3\} & & \\ & \{h_1, & & & h_4\} & \\ & & \{h_2, & & & h_5\} \end{array}$$

(enclose) $$\begin{array}{cccccc} \{h_0, & & & & & h_5\} \\ & \{h_1, & & & h_4\} & \\ & & \{h_2, & h_3\} & & \end{array}$$

IX. A negative example

On his return from Israel, Erdős thought he had generalized the proof to the ω^m's for $m < \omega$, only to learn on visiting Specker that he had shown that was impossible.

Theorem 3 (Specker 1957 [7])

$\omega^3 \nrightarrow (\omega^3, 3)^2$ (hence $\omega^m \nrightarrow (\omega^m, 3)^2$ for all $3 \leq m < \omega$).

IX. A negative example

There was one unavoidable pattern for $[\omega]^3_<$ that does not *admit a trio*, i.e. a pattern for which no triple can be constructed so that each pair has that pattern:

$$\begin{array}{ccccccc} a & a & & a & & & \\ & & b & & b & b \end{array}$$

or

$$\begin{array}{cccc} A & & a & \\ & b & & B \end{array}$$

if A, B represent sequences of length at least 2.

X. Another trio

Here is an unavoidable pattern that admits a trio:

$$\begin{array}{ccccccccc}
A & & a & & A & & a & & A \\
 & B & & b & & b & & B &
\end{array}$$

And here is a sample trio:

$$\begin{array}{cccccccccccccccc}
A & & & & & & a & & & A & & & a & & & A \\
 & B & & b & & B & & b & & & & b & & & B & \\
 & & C & & c & & & & c & & c & & & C & &
\end{array}$$

if A, B represent segments of length at least 2.

XI. A trio question

Question 1

Is there a pattern made up of segments of at most three weights, say $\{A, a, \alpha\}$, $\{B, b, \beta\}$ such that

each sequence begins with a segment of largest weight (A, B)

each sequence ends with a segment of largest weight (A, B)

but the pattern does not admit a trio?

XII. An enduring problem

The question about patterns not admitting a trio relates to a question asked Erdős over the years.

Question 2 (Erdős)

For which countable ordinals α, does $\alpha \to (\alpha, 3)^2$? ($1,000 1987 [3])
(Schipperus called such ordinals *partition ordinals* in his 1999 thesis.)

Theorem 4 (Galvin 1975 [4])

If $\alpha = \omega^\beta$ and $\beta = \beta_0 + \beta_1$ is decomposable, then $\alpha \nrightarrow (\alpha, 3)^2$.

XIII. A look on the positive side

Theorem 5

(Chang 3 1972 [1]; Milner $m \geq 3$ unpublished; L. shorter proof 1972)

$$\omega^\omega \to (\omega^\omega, m)^2, \text{ for } m \geq 3.$$

(Schipperus 2010 [6]) If β is indecomposable or $\beta = \gamma + \delta$ where $\gamma \geq \delta \geq 1$ are indecomposable, then

$$\omega^{\omega^\beta} \to (\omega^{\omega^\beta}, 3)^2.$$

Darby (unpublished) proved $\omega^{\omega^2} \to (\omega^{\omega^2}, 3)^2$ about the same time as Schipperus proved his theorem.

XIV. More negative partition relations

Theorem 6

(Darby 6 1999 [2]; Schipperus 6 1999, 2010 [6]; Larson 5 2000 [5])
If $\beta \geq \gamma \geq 1$, then

$$\omega^{\omega^{\beta+\gamma}} \not\to (\omega^{\omega^{\beta+\gamma}}, 5)^2.$$

(Darby (1999); Schipperus (1999, 2010))
If $\beta \geq \gamma \geq \delta \geq 1$, then

$$\omega^{\omega^{\beta+\gamma+\delta}} \not\to (\omega^{\omega^{\beta+\gamma+\delta}}, 4)^2.$$

(Schipperus (1999, 2010))
If $\beta \geq \gamma \geq \delta \geq \varepsilon \geq 1$, then

$$\omega^{\omega^{\beta+\gamma+\delta+\varepsilon}} \not\to (\omega^{\omega^{\beta+\gamma+\delta+\varepsilon}}, 3)^2.$$

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XV. References

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A final remark