Descriptive Set Theory

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Polish spaces

Polish space = complete metric separable.

Examples

- $\mathbb{N} = \omega = \{0, 1, 2, \dots\}$.
- $\mathbb{R} =$ real numbers. (E.g., Dedekind cuts)
- 2^ω = "Cantor space" = all infinite 01-sequences
- ω^ω = "Baire space"
  = all infinite sequences of natural numbers.
- $\mathbb{R}^n$, $\mathbb{R}^\omega$, $(2^\omega)^\omega \simeq 2^\omega$.
- $\dots$ many others

Polish spaces: convergence

Polish space X = complete metric separable. To understand these spaces, we need to understand *convergence* in these spaces.

- $a_1 = (5, 5, 5, 5, 5, 5 \ldots)$
- $a_2 = (\mathbf{3}, 8, 0, 0, 0, 0, \ldots)$
- $a_3 = (\mathbf{3}, 6, 0, 5, 0, 0, \ldots)$
- $a_3 = (\mathbf{3}, \mathbf{1}, \mathbf{4}, 6, 0, 0, \ldots)$
- $a_9 = (\mathbf{3}, \mathbf{1}, \mathbf{4}, \mathbf{1}, 11, 1, \ldots)$
- ...
- $a_\omega = (3, 1, 4, 1, 5, 9, 2, 6, \ldots)$.

$C \subseteq X$ is *closed*, if convergent sequences in C have limit in C.
$U \subseteq X$ is *open* iff $X \setminus U$ is closed.

Continuum Hypothesis

Cantor's Continuum Hypothesis (CH): Every infinite subset of the reals is either countable or equinumerous with $\mathbb{R}$ (or with 2^ω, i.e., contains a 1-1 copy of 2^ω).
(Hilbert's problem list, 1900: first problem.)

The continuum hypothesis is neither provable nor refutable (from the usual axioms of set theory).
(Cantor thought he had a proof.)

Perfect sets

X=Polish, e.g. $\mathbb{R}$ or 2^ω.
A nonempty set $P \subseteq X$ is perfect if

- P is closed
- P has no isolated points. (Every point in P is a limit of a sequence of distinct points in P.)

Examples:

- $[0, 1] \subseteq \mathbb{R}$
- $C \subseteq [0, 1]$, the Cantor set.
- $2^\omega \simeq C$.

Fact

$S \subseteq X$ contains a perfect set iff S contains a Cantor set, i.e., iff there is a continuous injective map from 2^ω into S.
Hence: If X contains a perfect set, then X is uncountable.

Fact

$S \subseteq X$ contains a perfect set iff S contains a Cantor set, i.e., iff there is a continuous injective map from 2^ω into S.

Corollary

If X contains a nonempty perfect set, then X is uncountable.

Question

Is the converse true?

Effective continuum hypothesis: Every infinite subset of the reals is either countable or contains a perfect set, i.e., contains a 1-1 *continuous* copy of 2^ω.

Effective Continuum Hypothesis

Effective continuum hypothesis: Every infinite subset of the reals is either countable or contains a perfect set, i.e., contains a 1-1 *continuous* copy of 2^ω.

The effective continuum hypothesis is (strictly speaking) false, but in practice often true.

- Refutable in ZFC. But the counterexample is "complicated".
- Not refutable in ZF without the axiom of choice. (disclaimer. . .)
- Provable for many definable sets in ZFC, even in ZF.
- Provable for most definable sets in "ZFC+large cardinals".

Thesis: Never claim that a set is merely uncountable when you can in fact show that it contains a perfect set.

Perfect set property

A class $\mathcal{C}$ of subsets of X has the *perfect set property* if each uncountable set in $\mathcal{C}$ contains a perfect set. (I.e., if all sets in $\mathcal{C}$ satisfy the effective CH.)

EXAMPLES:

- open sets
- closed sets
- Borel sets
- analytic sets

Note: being countable is a positive property. Being uncountable is a *negative* property. But "contains a perfect set" is positive!

Borel sets

$$
\begin{array}{ccccc}
 & & \vdots & & \\
\Pi^0_\omega & & & & \Sigma^0_\omega \\
 & \nwarrow & & \nearrow & \\
 & & \Delta^0_\omega & & \\
\vdots & & \vdots & & \vdots \\
G_\delta = \Pi^0_2 & & & & \Sigma^0_2 = F_\sigma \\
 & \nwarrow & & \nearrow & \\
 & & \Delta^0_2 & & \\
 & \nearrow & & \nwarrow & \\
\text{closed} = \Pi^0_1 & & & & \Sigma^0_1 = \text{open} \\
 & \nwarrow & & \nearrow & \\
 & & \Delta^0_1 & &
\end{array}
$$

Borel sets, examples

We consider subsets of 2^ω.

- $\{x \in 2^\omega : x(2) = 0 \ \& \ x(3) = 1\}$ is clopen, i.e., Δ^0_1.
- The set of all "eventually zero" sequences is countable, hence F_σ, i.e., Σ^0_2.
- The set U of all "uniformly distributed sequences" is Π^0_3:

$$x \in U \ \Leftrightarrow \ \forall k \, \exists m \forall n \geq m : \left| \frac{\sum_{i=0}^{n} x(i)}{n} - \frac{1}{2} \right| \leq \frac{1}{k}$$

$$U = \bigcap_{k \in \omega} \bigcup_{m \in \omega} \bigcap_{m \geq n} \Delta^1_0 = \bigcap_{k \in \omega} \bigcup_{m \in \omega} \Pi^0_1 = \bigcap_{k \in \omega} \Sigma^0_2 = \Pi^0_3$$

The perfect set property for closed sets

Theorem (Cantor-Bendixson)

Let X be a closed uncountable set. (Subset of $\mathbb{R}^n$, of 2^ω, etc.) Then there is a perfect set P such that $C := X \setminus P$ is countable.

Proof sketch.

Let C_0 be the set of isolated points of X. C_0 must be countable, and $X^{(1)} := X \setminus C_0$ is still closed uncountable.
Continue for infinitely many steps: $X \supseteq X^{(1)} \supseteq X^{(2)} \supseteq \cdots$. Stop when $C_n = \emptyset$ (then $X^{(n)}$ is perfect).
If you are not done after ω steps, let $X^{(\omega)} := \bigcap X^{(n)}$, and continue.
This process (of taking “Cantor-Bendixson derivatives”) must stop at a countable ordinal.

Analytic sets

An analogy:

- Borel sets have a flavour that is similar to recursive (=decidable) subsets of the natural numbers. (E.g.: closed under complements.)
- Analytic sets correspond to r.e. sets (recursively enumerable, computably enumerable, semi-decidable). (**Not** closed under complements.)

Another analogy:

- Borel sets have a flavour that is similar to sets in **P** = polynomially decidable sets.
- Analytic sets correspond to sets in **NP**.

Analytic sets, several definitions

Let X be a Polish space ($\mathbb{R}^n$, ω^ω, etc). Let $S \subseteq X$ be nonempty. Then the following are equivalent:

1. There is a continuous $F : \omega^\omega \to X$ with $F[\omega^\omega] = S$.
2. There is a continuous $F : \omega^\omega \to X$ and a closed set $C \subseteq \omega^\omega$ with $F[C] = S$.
3. As above, but C may be a Borel set.
4. There is a closed (or Borel) subset of $\omega^\omega \times X$ whose projection to X is exactly S.

Analytic sets, THE definition

Let X be a Polish space. $S \subseteq X$ is analytic if $S = \emptyset$ or there is a continuous map $F : \omega^\omega \to X$ with $F[\omega^\omega] = S$.

(We should check X itself is analytic under this definition.)

Rule of thumb: Anything involving only $\exists k \in \omega$, $\forall k \in \omega$, $\bigcap_{k\in\omega}$, $\bigcup_{k\in\omega}$ is a Borel set.
If you have to use $\exists y \in \omega^\omega$ or $\exists r \in \mathbb{R}$, then analytic and usually not Borel.

Example

$\{x \in \omega^\omega \mid \exists(n_1 < n_2 < \cdots) : x_{n_1} | x_{n_2} | \cdots\}$ is analytic, not Borel.
$\{x \in \omega^\omega \mid \exists(n_1 < n_2 < \cdots) : x_{n_1} = x_{n_2} = \cdots\}$ is Borel.

Closure properties

The family of analytic sets contains all closed sets and is closed under

- countable unions (hence contains all closed sets),
- countable intersections (hence contains all Borel sets),
- continuous images (by definition),
- continuous preimages.
- But NOT under complement.

Empirical fact:
(100-ε)% of all sets considered in analysis are Borel sets.
(100-ε)% of the remaining ε% are analytic (or co-analytic).

The perfect set property for analytic sets

Let $F : \omega^\omega \to X$ be continuous, and assume that $A := F[\omega^\omega]$ is uncountable. We claim that A contains a perfect set. More precisely: we will find a copy of 2^ω in ω^ω on which F is 1-1.

For $s \in \omega^{<\omega}$, we write $[s]$ for the $\{x : x \in \omega^\omega, s \subseteq x\}$ (= all branches extending the node s.)
For $s \in \omega^{<\omega}$, we write $F[s]$ for the set $\{f(x) : x \in \omega^\omega, s \subseteq x\}$.

We will show our claim under the additional assumption that $F[s]$ has more than one element, for each s.
(This can easily be arranged: Iteratively remove those s for which this is not true; with each s we remove from ω^ω all elements extending s, but all of them have the same image. Since only countably many s were removed, only countably many elements from A were removed.)

The perfect set property for analytic sets, part 2

Assumption: $F : \omega^\omega \to X$ is continuous, and for each $s \in \omega^{<\omega}$ the set $F[s] := \{f(x) : x \in \omega^\omega, s \subseteq f\}$ has > 1 elements.
(NOTE: for different s, the sets $F[s]$ are not necessarily disjoint. If they were, everything would be trivial.)

Construction:

1. Let $t := \langle\rangle$, the empty sequence.
2. We find two disjoint nonempty open sets U_0, U_1 in $F[t]$.
3. We find $t_0, t_1 \in \omega^{<\omega}$, extending t, such that $F[t_\ell] \subseteq U_\ell$. (Use continuity of F.)
4. Continue by splitting t_0 to t_{00} and t_{01}, and t_1 to t_{10}, t_{11}, using disjoint open sets U_{00}, U_{01}, U_{10}, U_{11}.
5. etc.

The perfect set property for analytic sets, part 3

What we have: For each finite $0, 1$-sequence s we have found a finite sequence $t_s \in \omega^{<\omega}$ such that:

- If s' extends s, then $t_{s'}$ extends t_s
- If s and s' are incompatible, then also t_s and $t_{s'}$ are incompatible, and moreover: $F[t_s]$ and $F[t_{s'}$ are disjoint.

What we want: A continuous 1-1 map from 2^ω into $F[\omega^\omega]$.

Easy: For each $x \in 2^\omega$, say $x = (a, b, c, \ldots)$, let t_x be the limit of t, t_a, t_{ab}, t_{abc}.
The map $x \mapsto F(t_x)$ is 1-1 and continuous.

Lebesgue measure

Every open subset U of $\mathbb{R}$ is a union of (finitely or countably many) disjoint intervals.
The measure $\mu(U)$ of U is the sum of the lengths of these intervals.
The measure μ is defined on all Borel sets, and it is σ-additive.

A set $N \subseteq \mathbb{R}$ has measure zero if for all ε there exists an open set U of measure $< \varepsilon$ with $N \subseteq U$.

A set $S \subseteq \mathbb{R}$ is measurable iff there exists a Borel (or Σ^0_2) set B such that $B \Delta S$ has measure 0. We set $\mu(S) := \mu(B)$.

Measure, continued

Theorem

All analytic sets are measurable.

Definition

Let X, Y be measure spaces. A function $g : X \to Y$ is measurable if $g^{-1}(U)$ is measurable for all open sets $U \subseteq Y$.

Measurable functions from some unspecified space X into "our" space Y are called "random variables" in statistics.

Uniformisation

Theorem (AC)

Let $S \subseteq X \times Y$, and assume $\pi S = X$, i.e.,

$$\forall x \in X : S_x := \{y \in Y \mid (x, y) \in S\} \neq \emptyset.$$

Then there is a function $g : X \to Y$ whose graph is contained in S, i.e., $g(x) \in S_x$ for all x.

(This theorem is equivalent to the axiom of choice.)

Theorem ("Uniformisation theorem")

Assume that S from above is nice. Then we can find a nice g. (Usually without the axiom of choice.)

Jankov-von Neumann

Theorem

Let $S \subseteq X \times Y$ be analytic, $\pi S = X$.
Then there is a measurable function $g \subseteq S$.

For the proof, we may assume that S is closed: Let $S = F[\omega^\omega]$, and

$$\hat{S} := \{(x, y, z) \in X \times Y \times \omega^\omega \mid (x, y) = F(z)\} \subseteq X \times (Y \times \omega^\omega)$$

If we can find a function $\hat{g} : X \to Y \times \omega^\omega$ uniformizing $\hat{S}$, then we get g uniformizing S by projection.
(Details omitted.)

Closed subsets of ω^ω

Let $C \subseteq \omega^\omega$.
The set $T_C := \{c \restriction n : c \in C, n \in \omega\} \subseteq \omega^{<\omega}$ is a tree. (Downward closed set of finite sequences.)
(Moreover, T_C has no leaves = nodes without extensions.)
We write $[T_C]$ for the set of its branches:

$$[T_C] := \{x \in \omega^\omega : \forall n \, x \restriction n \in T_C\} \supseteq C.$$

It is easy to check that $[T_C] =$ closure of C.

Hence: Closed sets = trees without leaves.

Jankov-von Neumann for closed sets

Theorem

Let $C \subseteq \omega^\omega \times \omega^\omega$ be closed, $\pi C = X$.
Then there is a nice (in fact: measurable) function $g \subseteq C$.

Proof sketch.

For each $x \in \omega^\omega$, the set $C_x := \{y : (x, y) \in C\}$ is closed, so $C_x = [T_x]$ for some tree T_x.
Let $g(x) :=$ the leftmost branch of $[T_x]$.
This is a nicely defined function, hence measurable. (Need to check details.)

An application

Fact

Let $(B_n : n \in \omega)$ be a family of sets of measure zero.
Then $\bigcup_n B_n$ has measure zero. (Trivially.)
But: There are families $(B_x : x \in \omega^\omega)$ of measure zero sets such that in $\bigcup_x B_x$ does not have measure zero. (Easy to find.)

Theorem

Let $(B_x : x \in \omega^\omega)$ be a nice family of sets $\subseteq Y$ with the following property: Whenever $x_1 \leq x_2$, then $B_{x_1} \subseteq B_{x_2}$.
Then the set $\bigcup_{x \in \omega^\omega} B_x$ still has measure zero.

("nice" means that the set $\{(x, y) : y \in B_x\}$ is a Borel (or analytic) set. By $x_1 \leq x_2$ I means that $x_1(n) \leq x_2(n)$ for all n.)