Basic set-theoretic techniques in logic
Part V, Infinite Games

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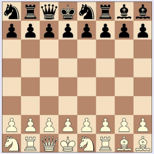
Let's play chess...

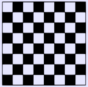
Ernst Zermelo (1871–1953)



Ernst Zermelo, Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels,

What is a chess configuration?

- There are 64 squares on a chess board:



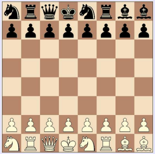
- There are 13 possible ways to fill a square.
  X ♙♟ ♖ ♜ ♘ ♞ ♗♝ ♕♛ ♔ ♚
- Thus, there are at most 64^{13} possible configurations of chess. Most of them are illegal.

Chess as a mathematical object (1).

We can think about playing chess as playing a sequence of natural numbers that stand for these at most 64^{13} configurations:

$$n_0 \xrightarrow{\text{WHITE}} n_1 \xrightarrow{\text{BLACK}} n_2 \xrightarrow{\text{WHITE}} n_3 \rightarrow \ldots$$

where n_0 corresponds to



and each of the other numbers represents something like

or

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Chess as a mathematical object (2).

We can think of the entire game tree to be the infinite 64^{13}-branching tree (i.e., a finitely branching tree). There are a number of different types of nodes in this tree:

1. Nodes that end in an illegal position,
2. nodes in which WHITE has lost,
3. nodes in which BLACK has lost,
4. nodes that determine that the game is a draw,
5. nodes in which neither of the following cases has occurred.

If the same configuration occurs twice for the same player, then a game is counted as a draw. So, any sequence of $2 \cdot 64^{13} + 1$ moves in chess in which neither of 1., 2., or 3. has occurred is a draw. That means that we can cut off the tree at $2 \cdot 64^{13} + 1$ and obtain a finite tree.

Chess as a mathematical object (3).

Let's prune the tree:

1. Nodes that end in an illegal position,
2. nodes in which WHITE has lost,
3. nodes in which BLACK has lost,
4. nodes that determine that the game is a draw,
5. nodes in which neither of the following cases has occurred.

Step 1. If the last position of a node p is an illegal position, search backwards to the root and find the first position p^* in that sequence that is illegal. Cut off the tree after that node.

Step 2. If p is a node in which WHITE or BLACK has lost or the game is a draw, cut off the tree after that node.

In the resulting tree T, the terminal nodes are exactly those in which it is determined whether WHITE won, BLACK won or the game is a draw.

Chess as a mathematical object (4).

Define $\mathrm{depth}(p)$ to be the length of the longest path from p to a terminal node. Note that for every $p \in T$, $\mathrm{depth}(p) \leq 2 \cdot 64^{13} + 1$. Note furthermore that a node p is terminal if and only if $\mathrm{depth}(p) = 0$.

Define a function label by recursion:

- If p is terminal, and t is a loss for WHITE, then let $\mathrm{label}(p) = \text{BLACK}$.
- If p is terminal, and t is a loss for BLACK, then let $\mathrm{label}(p) = \text{WHITE}$.
- If p is terminal, and t is a draw, then let $\mathrm{label}(p) = \text{DRAW}$.
- If p is terminal, and the last position is illegal, then if the last move was for WHITE, then let $\mathrm{label}(p) = \text{BLACK}$; if the last move was for BLACK, then let $\mathrm{label}(p) = \text{WHITE}$.

This defines label on all nodes of depth 0. The label determines the outcome of the game if the game reaches that node.

Chess as a mathematical object (5).

Suppose label is define for all nodes of depth i. Let p be a node of depth $i+1$ where WHITE has to move. All successors of p already have labels.

Case 1. If at least one of them is labelled WHITE, then label p WHITE as well.

Case 2. If none of them is labelled WHITE, but at least one is labelled DRAW, then label p DRAW.

Case 3. If all of them are labelled BLACK, then label p BLACK.

Now let p be a node of depth $i+1$ where BLACK has to move.

Case 1. If at least one of them is labelled BLACK, then label p BLACK as well.

Case 2. If none of them is labelled BLACK, but at least one is labelled DRAW, then label p DRAW.

Case 3. If all of them are labelled WHITE, then label p WHITE.

By the recursion principle, label is a total function on T, and thus the root has a label.

Chess as a mathematical object (6).

Theorem. If the root has label WHITE, then WHITE has a winning strategy; if the root has label DRAW, then both players have a drawing strategy; if the root has label BLACK, then BLACK has a winning strategy.

Corollary. One of the following three cases holds:

1. WHITE has a winning strategy in chess,
2. BLACK has a winning strategy in chess,
3. both players have a drawing strategy in chess.

Of course, to this day, it is not known which of the three cases holds.

Infinite games.

We fix an arbitrary set X of possible moves. We have two players, I and II. I plays in the even rounds (0,2,4,...) and II plays in the odd rounds (1,3,5,7,...).

Together, they produce an infinite sequence

$$x_0, x_1, x_2, x_3, x_4, \ldots$$

i.e., a function $x : \mathbb{N} \to X$.

We fix a payoff function $A : X^{\mathbb{N}} \to \{\mathrm{I}, \mathrm{II}, \textsc{draw}\}$.

Combinatorially, think of this as the infinitely long X-branching tree T_X in which the two players move by alternatingly producing an infinite branch.

Chess as a special case.

Let $X = 64^{13}$, and consider the finite pruned tree we constructed before as $T_{\mathrm{Chess}} \subseteq T_X$. Suppose that x is an infinite branch through T_X. Then it passes through a unique terminal node of T_{Chess}.

Now define

$$A_{\mathrm{Chess}}(x) := \mathrm{label}(t_x).$$

Some simplifying conventions.

From now on, we'll let $X = \mathbb{N}$, and we'll ignore the option DRAW. That means in our games, exactly one of the players wins.

This means that we do not really need a payoff function anymore, but can instead use a payoff set $A \subseteq \mathbb{N}^{\mathbb{N}}$, interpreting an outcome

$$x \in A$$

as a win for player I and an outcome

$$x \notin A$$

as a win for player II.

Strategies.

Let T^{I} be the set of nodes of $T_X = T_{\mathbb{N}}$ of even length; in other words, those nodes where player I has to play. Similarly, let T^{II} be the set of nodes of $T_X = T_{\mathbb{N}}$ of odd length.

A strategy for player I is a function $\sigma : T^{\mathrm{I}} \to \mathbb{N}$, and a strategy for player II is a function $\tau : T^{\mathrm{II}} \to \mathbb{N}$.

If σ and τ are such strategies, we can let them play against each other and recursively define $\sigma * \tau$:

$$(\sigma * \tau)(2n) := \sigma((\sigma * \tau)\restriction 2n)$$
$$(\sigma * \tau)(2n+1) := \tau((\sigma * \tau)\restriction 2n+1)$$

A strategy σ for player I is winning if for all τ, we have $\sigma * \tau \in A$.

A strategy τ for player II is winning if for all σ, we have $\sigma * \tau \notin A$.

Strategies as trees (1).

A strategy σ defines a tree $T_\sigma \subseteq T_{\mathbb{N}}$ by the following recursive definition:

- if $s \in T^{\mathrm{I}} \cap T_\sigma$, then $s\sigma(s) \in T_\sigma$;
- if $s \in T^{\mathrm{II}} \cap T_\sigma$, then $sx \in T_\sigma$ for any $x \in \mathbb{N}$.

If τ is any strategy for player II, then $\sigma * \tau$ is a branch through T_σ.

We can now reformulate: σ is winning for I if every branch through T_σ is in A.

Let's investigate T_σ. Let Z_σ be its set of branches. We'll show that $|Z_\sigma| = \mathfrak{c}$.

Strategies as trees (2).

Proof. It's easy to see that $|\mathbb{N}^{\mathbb{N}}| = \mathfrak{c}$. Since $Z_\sigma \subseteq \mathbb{N}^{\mathbb{N}}$, we get $|Z_\sigma| \leq \mathfrak{c}$.

For the other direction, we only need to produce an injection from the power set of $\mathbb{N}$ to Z_σ. As before, we identify the power set of $\mathbb{N}$ with $\{0,1\}^{\mathbb{N}}$ by

$$M \mapsto x_M$$

with

$$x_M(n) = \begin{cases} 1 & \text{if } n \in M, \\ 0 & \text{otherwise.} \end{cases}$$

We define a strategy for player II as follows. If $s \in T^{\mathrm{II}}$ and the length of s is $2n+1$, we let

$$\tau_M(s) := x_M(n).$$

Now consider $\sigma * \tau_M$. Clearly, if $M \neq M'$, then $\sigma * \tau_M \neq \sigma * \tau_{M'}$. So,

$$M \mapsto \sigma * \tau_M$$

is an injection from the power set of $\mathbb{N}$ into Z_σ. q.e.d.

Strategies as trees (3).

Using the same technique, you can show that there are exactly $\mathfrak{c}$ many strategies.

Proof. Homework.

Corollary.

1. If A is countable, then I cannot have a winning strategy in the game with payoff set A.
2. If the complement of A is countable, then II cannot have a winning strategy in the game with payoff set A.

An application (?)

Theorem (Morton Davis). For each $A \subseteq \mathbb{N}^{\mathbb{N}}$ there is a game G^*_A such that

1. If I has a winning strategy in G^*_A, then $|A| = \mathfrak{c}$.
2. If II has a winning strategy in G^*_A, then $|A| \leq \aleph_0$.

Corollary. If we can show that all games have a winning strategy for one of the two players, then the Continuum Hypothesis holds.

Existence of non-determined sets.

Theorem. The Axiom of Choice implies the existence of a set such that neither of the players has a winning strategy.

Proof. We had seen that there are $\mathfrak{c}$ many strategies. Use the Axiom of Choice to list them in a wellordered list $\{\sigma_\alpha\ ;\ \alpha < \mathfrak{c}\}$. We also saw that for each of these strategies σ_α, its set of branches Z_α has $\mathfrak{c}$ many elements.

Recursively define disjoint sets A and B such that each strategy contains an element of A and B. Then there can be no winning strategy for either play in the game with payoff set A. q.e.d.

Gale-Stewart theorem.

A set A is called finite horizon if there is a set $W \in T_{\mathbb{N}}$ such that

$$x \in A \text{ if and only if } \exists p \in W(x \text{ passes through } p).$$

Theorem. For every finite horizon game there is either a winning strategy for player I or for player II.

Isn't this just like in the chess example?

> *You prune the tree after the nodes $p \in W$, then these nodes become terminal nodes, and you label them with I. Then you run the recursion and if the root gets label I, then I has a winning strategy; if not then II has a winning strategy.*

Well, it's not so simple since we now have infinitely branching trees.