Basic set-theoretic techniques in logic
Part IV: The Axiom of Choice

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Why axioms?

If we start playing with sets without any care we are in trouble:

The Russell paradox

Suppose that there is a set X such that

$$X = \{a : a \notin a\}.$$

Then $X \in X \iff X \notin X$, which is a false sentence.

Therefore we start by a list of axioms naming some legitimate operations.
The Russell paradox show that it is not always possible to form a set of the form $\{x : \varphi(x)\}$.

Some axioms of Zermelo-Fraenkel (ZF)

- **Union:** For any sets X and Y there is a set, denoted $X \cup Y$, containing all elements of X and all elements of Y.
- **Separation:** If A is a set and φ is some property then there is a set $\{x \in A : \varphi(x)\}$.
- **Power set:** For every set X there is a set $\{A : A \subseteq X\}$ (denoted $P(X)$).
- **Infinity:** There is an infinite set.

The axiom of choice (AC)

For every family $\mathcal{A}$ of nonempty sets there is a choice function f, such that $f(A) \in A$ for every $A \in \mathcal{A}$.

ZF+ AC=ZFC.

Equivalent forms

1. **The axiom of choice.**
2. **Zermelo's theorem:** Every set can be well-ordered.

Proof.

(1)$\longrightarrow$(2) Take any set X. We shall show that $X = \{x_\alpha : \alpha < \gamma\}$ for some ordinal number γ.
Let f be a choice function for the family of all nonempty subsets of X. We define

$$x_\alpha = f(X \setminus \{x_\beta : \beta < \alpha\}),$$

until it is possible. Then take γ to be the first ordinal number for which the set $X \setminus \{x_\alpha : \alpha < \gamma\}$ is empty.
(2)$\longrightarrow$(1) If $\mathcal{A}$ is any family of nonempty sets then we can order the union $X = \bigcup \mathcal{A}$ of all of them and define $f(A)$ to be the first element of A. □

Zorn's lemma, Tuckey style

Theorem

Let X be a set and $\mathcal{A}$ be a family of its subsets. Assume that $\mathcal{A}$ has finite character, i.e. $B \in \mathcal{A}$ if and only if all finite subsets of B belong to $\mathcal{A}$.
Then for any $A \in \mathcal{A}$ there is $M \in \mathcal{A}$ such that $A \subseteq M$ and M is maximal, i.e. for every $M' \in \mathcal{A}$ satisfying $M \subseteq M'$ we have $M' = M$.

Proof.

Let $X = \{x_\alpha : \alpha < \gamma\}$. Define M by

$$x_\alpha \in M \Leftrightarrow A \cup \{x_\beta \in M : \beta < \alpha\} \cup \{x_\alpha\} \in \mathcal{A}.$$

Then $M \in \mathcal{A}$ because all the finite subsets of M are in $\mathcal{A}$. □

Application: Hamel basis

A set $\{x_1, x_2, \ldots, x_n\}$ of reals is linearly independent over $\mathbb{Q}$ if for any $q_i \in \mathbb{Q}$, if $q_1 x_1 + q_2 x_2 + \ldots + q_n x_n = 0$ then $q_i = 0$ for all $i \leq n$.

Example

$\{1, \sqrt{2}, \sqrt{3}\}$ is linearly independent over $\mathbb{Q}$.

Remark

If $\{x_1, x_2, \ldots, x_n\}$ is l.i. while $\{x_1, x_2, \ldots, x_n, y\}$ is not then $y = q_1 x_1 + \ldots q_n x_n$ for some q_i's.

Theorem

There is a maximal linearly independent over $\mathbb{Q}$ set $H \subseteq \mathbb{R}$. Every $x \in \mathbb{R}$ has the unique representation $x = \sum_{i \leq n} q_i h_i$, where $n \in \mathbb{N}$, $q_i \in \mathbb{Q}$, $h_i \in H$.

Application: Vitali sets

For $x, y \in \mathbb{R}$, say that $x \sim y$ if $x - y \in \mathbb{Q}$. The relation is equivalence relation on $\mathbb{R}$, that is $x \sim x$, $x \sim y \Leftrightarrow y \sim x$ and $x \sim y, y \sim z \Rightarrow x \sim z$ for any x, y, z. The relation $\sim$ divides $\mathbb{R}$ into disjoint nonempty sets, where each set is of the form $\{y : y \sim x\}$ for some x. Let V be a selector for that partition. Then

- $(q + V) \cap V = \emptyset$ for every rational $q \neq 0$; otherwise, if $x \in (q + V) \cap V$ then $x = y + q$ for some $x, y \in V$, which gives $x \sim y, x \neq y$, a contradiction.
- $\bigcup_{q\in\mathbb{Q}}(q + V) = \mathbb{R}$.
- We can assume that $V \subseteq [0, 1)$. Then

$$[0,1) \subseteq \bigcup_{q\in\mathbb{Q}\cap[-1,1)} (q + V) \subseteq [-1, 2).$$

Is Axiom of Choice controversial?

Banach-Tarski paradox

The ball of radius 1 (in $\mathbb{R}^3$) can be, by AC, decomposed into 5 pieces. Using those sets one can, using rotations and translations, form two balls of radius 1.

It follows that $1{=}2$ so there must be something wrong with AC.
...Or with you intuition concerning the volume. Why do you assume that you can measure the volume of every set in $\mathbb{R}^3$?

Thank yor for your attention!

David Hilbert:

No one shall expel us from the Paradise that Cantor has created.

Georg Cantor:

The essence of mathematics lies entirely in its freedom.