Basic set-theoretic techniques in logic
Part II, Counting beyond infinity: Ordinal numbers

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Summary of the first lecture:
We have discussed how to measure the infinity, in particular measuring the size of the set o natural numbers:

$$\aleph_0$$

Now for something slightly different. Have you ever counted up to 1000?

$$1, 2, 3, \ldots, 1000.$$

It takes more than 16 minutes but surely we can do it.
We can also imagine ourselves counting up to $10^{10^{10}}$ though it will be really time-consuming.
Can we count beyond infinity? If so we need a new name:

$$\omega$$

Having ω (sometimes denoted ω_0) at hand we can continue:

$$0, 1, 2, \ldots, 2011, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega, \ldots$$

Is there any use of this?

Pros and Cons of Hitch Hiking[a]

[a] after Roger Waters, ex Pink Floyd

- Suppose you are hitchhiking from A to B.
- Pros: should be for free.
- Cons: the route may be complicated,
- starting from $A \to B$ (0 stops), to
- $A \to S_1 \to B$ (1 stop),
- $A \to S_1 \to S_2 \to B$ (2 stops),
- and so on.

Pros and Cons of Hitch Hiking, continued

Suppose that on our way we may be offered a lift by little dwarfs in their little cars, moving us only a tiny little bit forward:

$$A \to S_1 \to S_2 \to \ldots \to S_n \to \ldots \to B.$$

Then we may call such a route ω. Can you imagine hitchhiking in the $\omega + 1$ or $\omega + \omega$ style?

All the hitchhiker's routes

Denote the collection of all possible routes by ω_1;

$$\omega_1 = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2 \ldots, \omega + \omega, \omega + \omega + 1, \ldots\}.$$

- Note that $1 + \omega$ is the same as ω but $\omega + 1$ is different.
- With every route α we can think of $\alpha + 1$, so there is no largest element of ω_1.
- Every route α has finitely or countably many stops.
- If α is a route and X is any nonempty set of stops appearing in α then X has the first stop.
- If $\alpha_1, \alpha_2, \ldots$ is any sequence of routes then there is a route α which is more complicated than all α_n's.
- The set ω_1 of all routes is uncountable.

Now a serious stuff!

Definition

We say that a set X is **linearly ordered** by $<$ if for any $x, y, z \in X$

- $x \not< x$;
- $x < y$ and $y < z$ imply $x < z$;
- if $x \neq y$ then $x < y$ or $y < x$.

Example

The set $\mathbb{R}$ of reals is linearly ordered by the 'natural' order.
All words are linearly ordered by the lexicographic order.

Definition

A set X is **well-ordered** by $<$ if it is linearly ordered and

- every nonempty subset A of X has a least element.

Example

The set $\mathbb{N}$ is well-ordered. Hmmmm, should be obvious...
The interval $[0, 1]$ has the least element $(= 0)$ but is not well-ordered because its subset $A = \{1, 1/2, 1/3, \dots\}$ does not contain a least element.

Definition

Two well-ordered sets $(X, <)$ and $(Y, <)$ are **isomorphic** if there is a bijection $f : X \to Y$ such that

- $x_1 < x_2$ is equivalent to $f(x_1) < f(x_2)$;

for any $x_1, x_2 \in X$.

Theorem

1. *If $(X, <)$ is well-ordered and $f : X \to X$ is an increasing function then $f(x) \geq x$ for every $x \in X$.*
2. *If $(X, <)$ is well-ordered and $f : X \to X$ is an isomorphism then f is the identity function.*

Proof.

Suppose that $f(x) \geq x$ does not hold for all x; it means that the set

$$A = \{x \in X : f(x) < x\}$$

is nonempty. Take its minimal element x_0. Then $y_0 = f(x_0) < x_0$ (since $x_0 \in A$), and $f(y_0) < f(x_0) = y_0$ (since f is increasing). It follows that $y_0 \in A$, a contradiction with $y_0 < x_0$.
By the first part we have $f(x) \geq x$ for any x. We can also apply the first part to the inverse function $f^{-1} : X \to X$:
$f^{-1}(x) \geq x$ so $x = f(f^{-1}(x)) \geq f(x)$.
Hence $f(x) = x$ for all x.

□

If X is well-ordered and $a \in X$ then the set $\{x \in X : x < a\}$ is called **the initial segment** of X given by a.

Theorem

Let $(X, <)$ and $(Y, <)$ be two well-ordered sets. Then either

1. *X and Y are isomorphic, or*
2. *X is isomorphic to some initial segment of Y, or*
3. *Y is isomorphic to some initial segment of X.*

Definition

An **ordinal number** is the order type of some well-ordered set.

If α is the order type of X and β is the order type of Y then

1. $\alpha = \beta$,
2. $\alpha < \beta$,
3. $\beta < \alpha$,

in the corresponding cases.

Example

- 0 is the order type of the empty set;
- 1 is the order type of a set consisting of one element;
- $\omega = \omega_0$ is the order type of $\{0, 1, 2, \ldots\}$;

We may as well think that ω **is** the set $\{0, 1, 2, \ldots\}$.

Definition

ω_1 is the least order type of a well-ordered uncountable set.

We have $\alpha < \omega_1$ whenever α is an order type of a countable set. We may think that $\omega_1 = \{0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \alpha, \ldots\}$ is the set of all order types of countable sets.

Ordinal and cardinal numbers

- An ordinal number α is a cardinal number if for every $\beta < \alpha$ we have $|\beta| < |\alpha|$.
- $0, 1, 2, \ldots$ are cardinal numbers.
- ω is a cardinal number (denoted $\aleph_0$).
- $\omega + 1$, $\omega + \omega$ are not cardinal numbers.
- ω_1 is the next cardinal number denoted as $\aleph_1$.
- ω_2 is the least order type of a set of cardinality $> \aleph_1$; $\aleph_2 = \omega_2$.
- We can define $\aleph_0 < \aleph_1 < \aleph_2 < \ldots$.
- Then $\aleph_\omega$ comes. **And so on ...** Do you understand?[a]

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[a] In mathematics, you don't understand things. You just get used to them. (John von Neumann)

Handling the continuum

- We have an exact list of cardinal numbers $\aleph_0 < \aleph_1 < \aleph_2 < \ldots$.
- Before we defined another list $\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < \ldots$.
- We also considered $\mathfrak{c}$ — the cardinality of $\mathbb{R}$.
- Let us prove that $\mathfrak{c} = 2^{\aleph_0}$.

$2^{\aleph_0} = \mathfrak{c}$.

Note that $2^{\aleph_0}$ (by the definition the cardinality of $P(\mathbb{N})$) is the cardinality of the set $\{0,1\}^{\mathbb{N}}$ of all infinite sequences of of 0's and 1's.

The function $f : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$, where

$$f(x_1, x_2, \ldots) = \sum_{n=1}^{\infty} \frac{2x_n}{3^n},$$

is one-to-one. It follows that $2^{\aleph_0} \leq \mathfrak{c}$.

Every $x \in [0,1]$ has a unique **infinite** binary expansion

$$x = (0, x_1 x_2 \ldots)_{(2)}.$$

This shows that $[0,1]$ admits one-to-one function into $\{0,1\}^{\mathbb{N}}$, and $\mathfrak{c} = |[0,1]| \leq 2^{\aleph_0}$. Finally $\mathfrak{c} = 2^{\aleph_0}$ by the Cantor-Bernstein theorem. □