Basic set-theoretic techniques in logic
Part I: Measuring the infinite

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In the real world...

...sets have finitely many elements, e.g.

- $W = \{\text{Mon}, \text{Tue}, \ldots, \text{Sat}\}$, $|W| = 7$;
- $EU = \{\text{Austria}, \text{Belgium}, \ldots, \text{United Kingdom}\}$, $|EU| = 27$;
- $UN = \{\text{Afghanistan}, \text{Albania}, \ldots, \text{Zimbabwe}\}$, $|UN| = 193$.

In mathematics ...

...many important sets are infinite, e.g.

- the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots, 2011, 2012, \ldots\}$;
- the set of rational numbers (all the quotients) $\mathbb{Q} = \{0, 1, 2, 2/3, 7/8, \ldots\}$;
- the set of all reals $\mathbb{R}$ (including $\mathbb{Q}$ and **many** other).

Although our world seems to be finite, we need the concept of infinity to describe it!
Saying that $\mathbb{N}$ and $\mathbb{R}$ are infinite, or writing $|\mathbb{N}| = |\mathbb{R}| = \infty$, is not enough.
In fact the symbol ∞ denotes rather *potential infinity*, e.g.

$$1 + 1/2 + 1/3 + \ldots = \infty,$$

while we shall discuss the *actual infinity*, of containing infinitely many elements. We shall see that infinity has many faces and we need several names for it.
It is intuitively clear that $\mathbb{N}$ is the 'smallest' infinite set.
It's infinite but countable, meaning: in theory, we can imagine naming all its elements.

Definition

We say that a set A is **countable** if we can write

$$A = \{a_1, a_2, \ldots, a_n, \ldots\}$$

where $a_1, a_2, \ldots$ are all distinct.

We can label all elements of a countable set A by natural numbers, so we think that A has the same number of elements as $\mathbb{N}$.

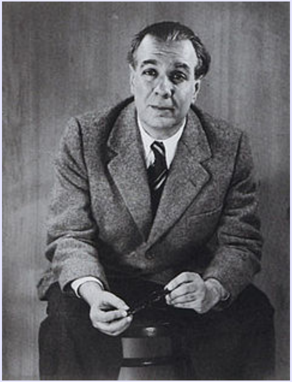
Aleph zero

The infinity represented by $\mathbb{N}$ is denoted by $\aleph_0$; we write

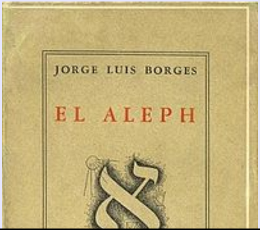
$$|\mathbb{N}| = \aleph_0.$$

Having introduced $\aleph_0$, we can write $|A| = \aleph_0$ instead of saying that A has as many elements as $\mathbb{N}$.

Why aleph? Why aleph with index 0? Why Borges?



Jorge Luis Borges (1899–1986)



Hotel $\aleph_0$

You are the owner of a hotel having inifnitely many rooms (numbered 1,2,...). Therefore if one day you have infinitely many guests $g_1, g_2, \ldots, g_n, \ldots$ then you can provide accomodation for all of them. Late in the evening another guest arrives? No problem:

$$g_1 \to 2, \quad g_2 \to 3, \quad \ldots, g_n \to n + 1.$$

You will have the room no 1 free for the late guest.
Next day you face another infinite group of tourists $h_1, h_2, \ldots, h_n, \ldots$. Still no problem:

$$g_1 \to 2, \quad g_2 \to 4, \ldots, g_n \to 2n \ldots.$$

This makes all the rooms with odd numbers free, and

$$h_1 \to 1, \quad h_2 \to 3, \ldots, h_n \to 2n - 1 \ldots.$$

Paradoxes of the infinite arise only when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited. (Galileo)

Paradoxes? Rather theorems: $\aleph_0 + 1 = \aleph_0$, $\aleph_0 + \aleph_0 = \aleph_0$.

Properties of countable sets

- If A and B are countable then $A \cup B$ is countable.
- If A and B are countable then $A \times B$ is countable, too, where

$$A \times B = \{\langle a, b\rangle : a \in A, b \in B\}.$$

- If $A_1, A_2, \ldots$ are all countable then the set

$$A = A_1 \cup A_2 \cup \ldots A_n \cup \ldots$$

containing all elements of all those sets, is countable too.

Theorem

The set $\mathbb{Q}$ of rational numbers is countable: $|\mathbb{Q}| = \aleph_0$.

Theorem

The set $\mathbb{R}$ of all real numbers is not countable.

Proof.

In fact we shall check that already the interval $[0, 1]$ is not countable.
Suppose that we have managed to create a list $a_1, a_2, \ldots$ of all real numbers $x \in [0, 1]$.

number	0.	1st	2nd	3rd	...	nth	...
a_1	0.	? x_1					
a_2	0.		? x_2				
a_3	0.			? x_3			
...	0.						
a_n	0.					? x_n	
...	0.						

The number $0.x_1x_2 \ldots x_n \ldots$ is not on our list!

Definition

The cardinality of $\mathbb{R}$ is called **continuum** and denoted by $\mathfrak{c}$:

$$|\mathbb{R}| = \mathfrak{c}.$$

Why not $\aleph_1$? Be patient!

Comparing arbitrary sets

- We say that two sets X and Y are **equinumerous** if there is a bijection $f : X \to Y$, that is one-to-one correspondence between all elements of X and all elements of Y.
- Equinumerous sets have the same cardinality: $|X| = |Y|$.
- Note that a set X is countable if it is equinumerous with $\mathbb{N}$.

Examples

- Every two nonempty intervals (a, b) and (c, d) on the real line are equinumerous and have cardinality $\mathfrak{c}$.
- **Theorem.** The plane $\mathbb{R} \times \mathbb{R}$ is equinumerous with $\mathbb{R}$.
- All the Euclidean spaces $\mathbb{R}^1, \mathbb{R}^2, \ldots, \mathbb{R}^d, \ldots$ have cardinality $\mathfrak{c}$.

Comparing arbitrary sets II

- $|X| \leq |Y|$ if there is a one-to-one function $f : X \to Y$, that is a bijection between X and some part of Y.
- $|X| < |Y|$ if $|X| \leq |Y|$ but $|X| \neq |Y|$.

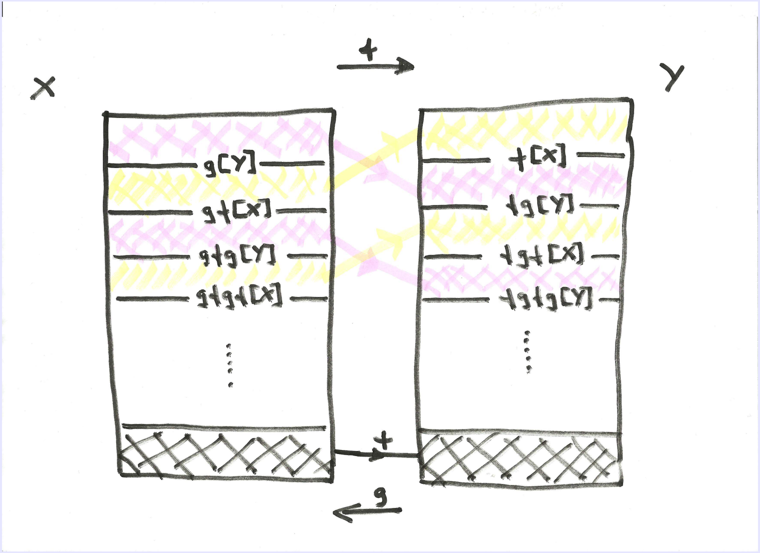
We already know that $|\mathbb{N}| < |\mathbb{R}|$, in other words: $\aleph_0 < \mathfrak{c}$.

Theorem (Cantor-Bernstein)

If $|X| \leq |Y|$ *and* $|Y| \leq |X|$ *then* $|X| = |Y|$.

Do not think it's obvious!

A math lecture without a proof is like a movie without a love scene. (H. Lenstra)



Definition

If X is any set we denote by $P(X)$ the power set of X, that is the family of all subsets of X.

Example

Let $X = \{1, 2, 3\}$. Then

$$P(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

If $X = \{1, 2, \ldots, n\}$ then $P(X)$ has 2^n elements.

Definition

If X is a set of cardinality κ then 2^κ denotes the cardinality of $P(X)$.

For a finite set X we have $2^{|X|} > |X|$ since $2^n > n$.

Theorem (Cantor)

For every set X the power set $P(X)$ has more elements than X; in other words

$$2^\kappa > \kappa,$$

for any cardinal number.

Proof.

We have $|X| \leq |P(X)|$ since we can define one-to-one function $f : X \to P(X)$ by $f(x) = \{x\}$.
Suppose that $g : X \to P(X)$ is a bijection. Consider the set $A \subseteq X$, where

$$A = \{x \in X : x \notin g(x)\}.$$

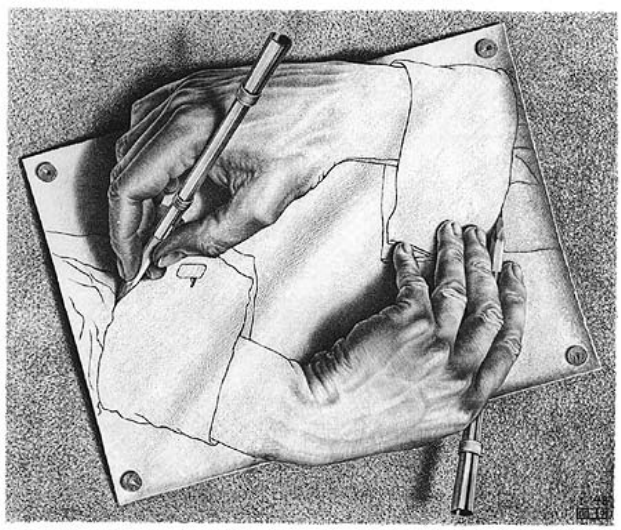
Then A cannot be associated with any $x \in X$. If we suppose that $A = g(x_0)$ then we have a puzzle whether x_0 is in A or not:

- if $x_0 \in A$ then $x_0 \notin g(x_0) = A$;
- if $x_0 \notin A$ then $x_0 \in g(x_0) = A$,

a contradiction!. □

Like in a famous **Barber paradox:**

In some village there was one man who was the only barber and he was ordered to shave all the men who do not shave themselves. Should he shave himself?



‘Drawing hands’ by Maurits Cornelis Escher

Conclusions

- $\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < \ldots$;
- there are **infinitely** many kinds of infinity;
- there is no set X which is the biggest one.

What about $\mathfrak{c}$? We shall see that

Theorem

$$\mathfrak{c} = 2^{\aleph_0}$$

We can also ask if there are only **countably** many types of infinity:-)

An application: Transcendental numbers

- Recall that $x \in \mathbb{R}$ is rational if $x = a/b$ for some integers a, b, $b \neq 0$.
- $\sqrt{2}$ is not rational but it solves the equation $x^2 - 2 = 0$.
- x is **algebraic** if x is a solution of some equation

  $$a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n = 0,$$

  for some integers a_i and some n.
- π and $e = \lim_{n\to\infty}(1 + 1/n)^n$ are not algebraic, they are **transcendental**. But it's difficult to prove it!
- Is there is an easy way of showing that there are transcendental numbers?
- The set of all algebraic numbers is countable; so a *typical number* is indeed transcendental.