Ordinals and Cardinals: Basic set-theoretic techniques in logic

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This course is a foundational course (no prerequisites) on basic techniques of set theory. It consists of five lectures (given by BL= 'Benedikt' and GP= 'Grzegorz'):

Programme

Monday	General introduction (BL)
	Measuring the infinite: Cardinal numbers (GP)
Tuesday	*Counting beyond infinity: Ordinal numbers* (GP)
Wednesday	*Transfinite recursion and induction* (BL)
Thursday	*The Axiom of choice* (GP)
Friday	*Set-theoretic analysis of infinite games* (BL)

No exams:-)

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The two protagonists of set theory.

- Cardinal numbers.

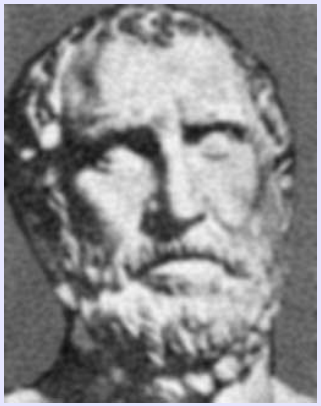
  Measuring the size of infinity and comparing the sizes of infinite sets.

- Ordinal numbers.

  Counting beyond infinity and providing the means of exhausting infinite sets by iterative processes.

Until the 19th century, infinity had been considered to be a rather problematic concept.

Achilles and the tortoise.

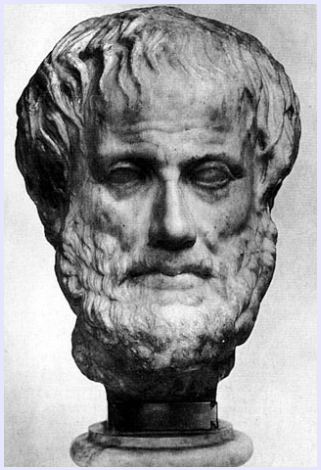


Zeno of Elea, c. 490 BC – c. 430 BC

The argument says that it is impossible for [Achilles] to overtake the tortoise when pursuing it. For in fact it is necessary that what is to overtake [something], before overtaking [it], first reach the limit from which what is fleeing set forth. In [the time in] which what is pursuing arrives at this, what is fleeing will advance a certain interval ... And in the time again in which what is pursuing will traverse this [interval] which what is fleeing advanced, in this time again what is fleeing will traverse some amount ...

Simplicius, On Aristotle's Physics, 1014.10

Aristotle and the Actual Infinite.



Aristotle, 384 BC - 322 BC

For generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different.

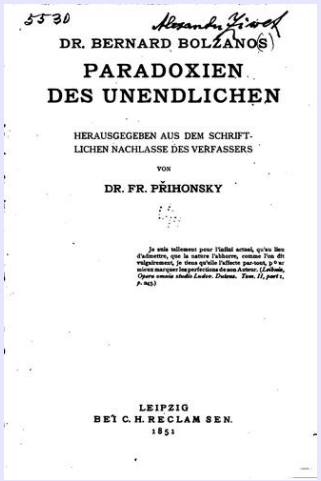
Aristotle, Physica, III.6

For the fact that the process of dividing never comes to an end ensures that this activity exists potentially, but not that the infinite exists separately.

Aristotle, Metaphysica, IX.6

Paradoxien des Unendlichen

Bernard Bolzano, 1781–1848



§18 Nicht eine jede Größe, die wir als Summe einer unendlichen Menge anderer, die alle endlich sind, ist selbst eine unendliche.

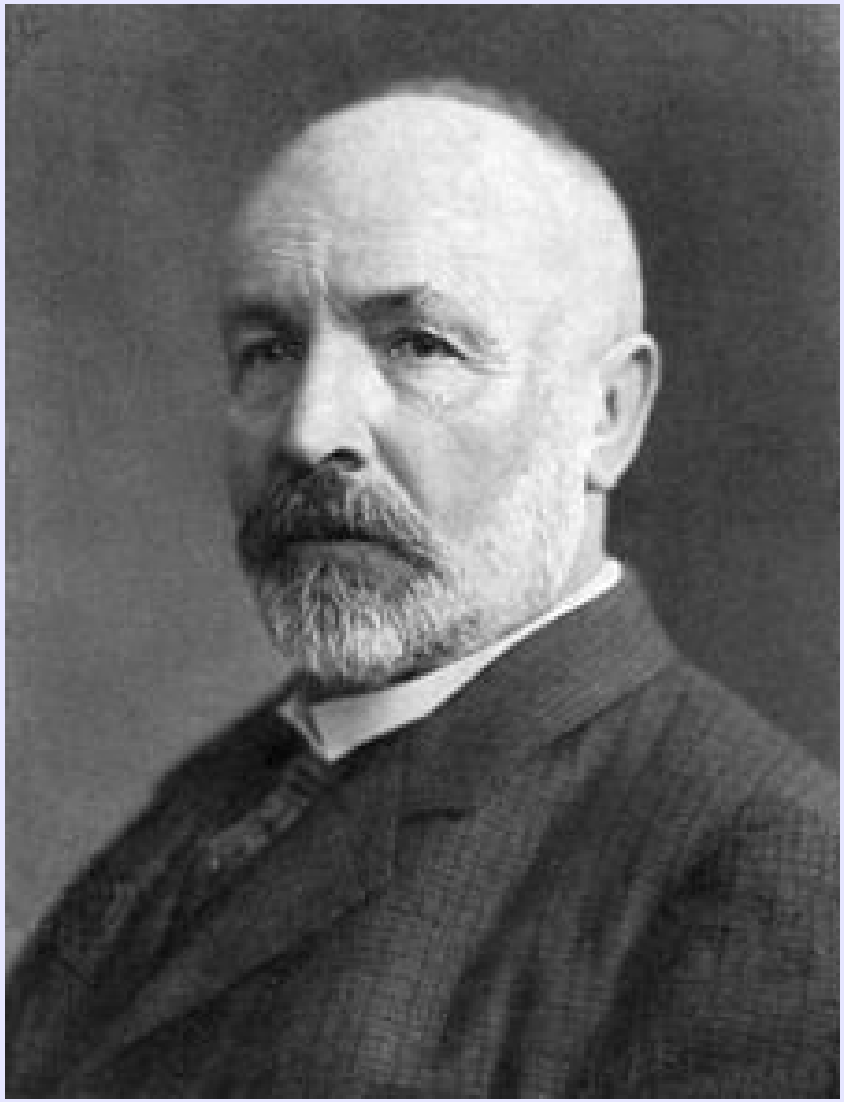
Not every magnitude that is a sum of infinitely many finite magnitudes is itself infinite.

§20 Ein merkwürdiges Verhältnis zweier unendlicher Mengen zueinander, bestehend darin, daß es möglich ist, jedes Ding der einen Menge mit dem der anderen so zu verbinden, daß kein einziges Ding in beiden Mengen ohne Verbindung bleibt, auch kein einziges in zwei oder mehr Paaren vorkommt.

A remarkable relationship between two infinite sets: it is possible to pair each object of the first set to one of the second such that every object in the two sets has a unique partner.

§21 Dennoch können beide unendliche Mengen, obschon mit Hinsicht auf die Vielheit ihrer Teile gleich, in einem Verhältnisse der Ungleichheit ihrer Vielheiten stehen, so daß die eine sich nur als ein Teil der anderen herausstellen kann.

And this situation can occur even if one of the sets is a proper subset of the other.



Georg Cantor (1845 – 1918)

- Cardinal numbers.

  Measuring the size of infinity and comparing the sizes of infinite sets.

- Ordinal numbers.

  Counting beyond infinity and providing the means of exhausting infinite sets by iterative processes.

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Cantor, Georg (1874). Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen, *Journal für die reine und angewandte Mathematik*, 77, 258-262.

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Cantor, Georg (1872). Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen, *Mathematische Annalen*, 5, 123–132.

The dual nature of set theory.

- Historically, set theory started as a field of mathematics: the study of infinite sets and their relationships.
- In subsequent years, set theory developed into more than that: the standard foundations for mathematics.

Ernst Zermelo
1871–1953

Abraham Fraenkel
1891–1965

Thoralf Skolem
1887–1963

John von Neumann
1903–1957

In our course, we shall ignore the foundational side of set theory, and rather discuss basic set theory as a technique to deal with infinities.

A recurring theme of this course will be the fact that you can exhaust infinite sets by a procedure called transfinite recursion. As an application, this will be used in our final lecture (Friday) to produce an algorithm to determine the winner of an infinite game.