The ghosts of departed quantities as the soul of computation

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FotFS8, Cambridge

[1]This research is generously supported by the John Templeton Foundation.

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Moreover: Infinitesimals and NSA are said to have 'non-constructive' nature (Bishop, Connes), although prominent in physics and engineering. The latter produces rather concrete/effective/constructive mathematics (compared to e.g. pure mathematics).

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Overarching question:

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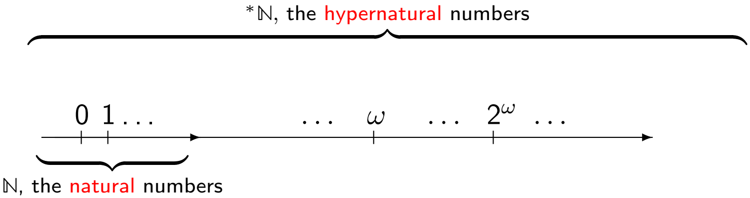
4. Other (SDG)

Nonstandard Analysis: a new way to compute



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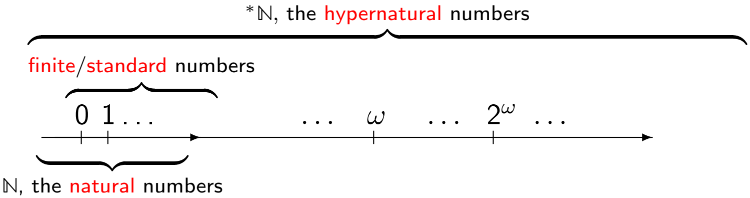

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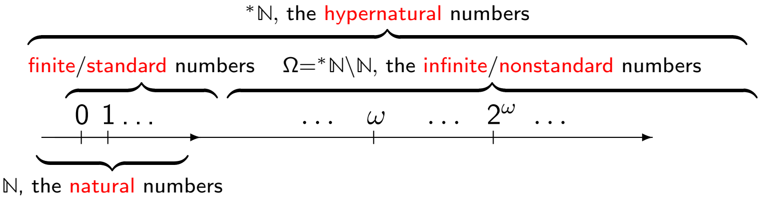

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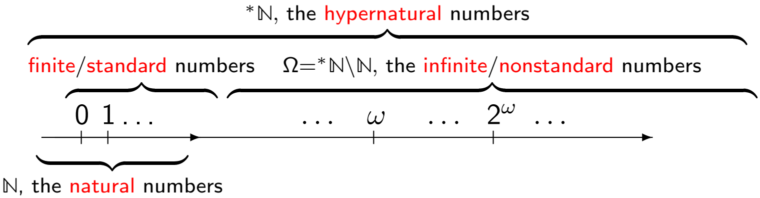

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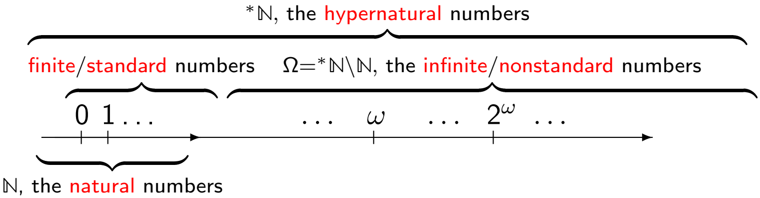

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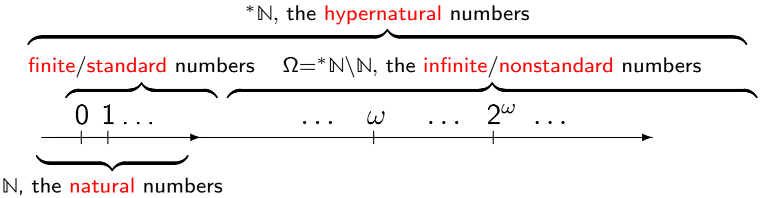

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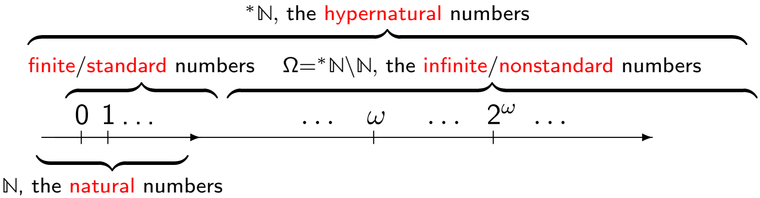

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Ω-invariant functions are nonstandard, i.e. 'come from above'.

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Classical existence of a standard object with the same standard and nonstandard properties = A standard functional computes the object.

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Hence, we can use Ω-CA to obtain a standard result from P'', without using the Halting problem.

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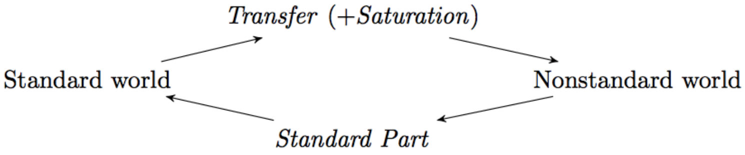

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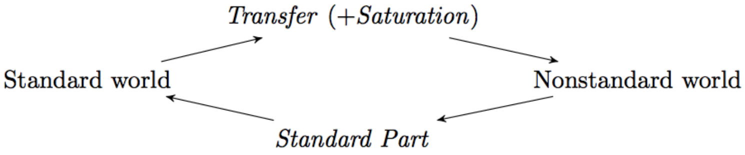

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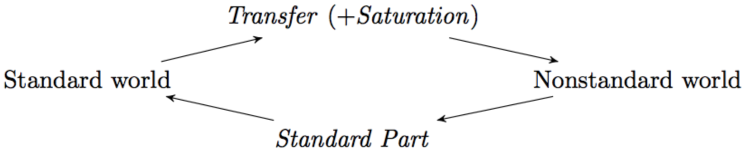

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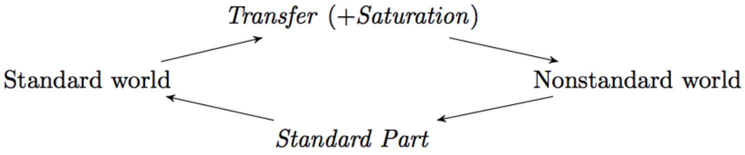

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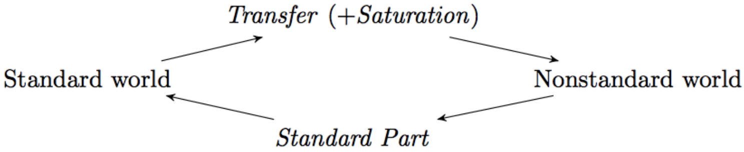

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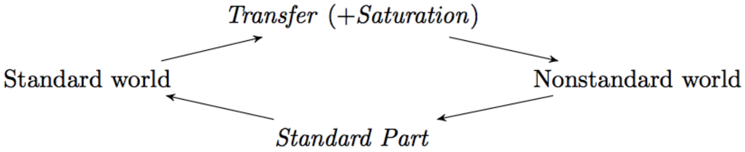

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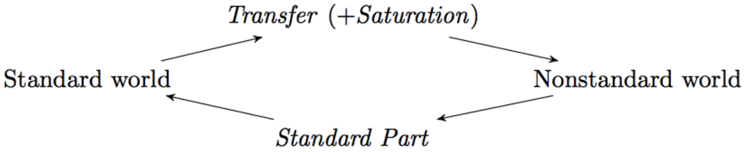

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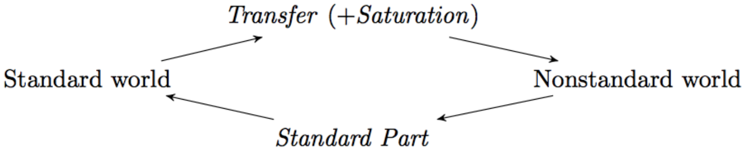

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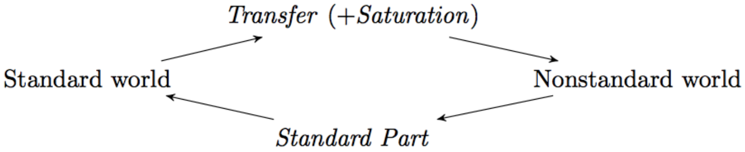

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To guarantee Ω-invariance, we need to assume 'constructive' definitions in the standard world.

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Any questions?