A True(r) History of Strict Finitism

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Confusion about what strict finitism is

- Both historical roots
- As what its meaning is

Different names:

- Strict finitism
- Ultrafinitism
- Ultra-intuitionism

Aim: to clarify matters (a bit)

The "founding father" (usually mentioned as such)

Alexander Yessenin-Volpin (sometimes Essenine-Volpin of Ésenine-Volpine)

- ultra-intuitionism
- articles quite cryptic
- no direct interest in finitism
- different aim (finitary consistency proof)

YESSENIN-VOLPIN, A. S. : "Le programme ultra-intuitioniste des fondements des mathématiques". In: Infinitistic Methods, Proceedings Symposium on Foundations of Mathematics, Pergamon Press, Oxford, 1961, pp. 201-223.

YESSENIN-VOLPIN, A. S. : "The ultra-intuitionistic criticism and the antitraditional program for foundations of mathematics". In: KINO, MYHILL & VESLEY (eds.), Intuitionism & proof theory. North-Holland, Amsterdam, 1970, pp. 3-45.

YESSENIN-VOLPIN, A. S. : "About infinity, finiteness and finitization". In RICHMAN, F. (ed.), 1981, pp. 274-313.

"Zenonian" sets Z:

- if n belongs to Z, so does n+1
- Z is nevertheless finite in its entirety

    example: the collection of heartbeats in your
        youth

But see: James R. Geiser: "A Formalization of
    Essenin-Volpin's Proof Theoretical Studies by
    Means of Nonstandard Analysis" (JSL, Vol. 39,
    No. 1, 1974, pp. 81-87)

for an attempt at rigorous reconstruction

"Take, for example, the unusual answer proposed by Alexander Yessenin-Volpin (Aleksandr Esenin-Volpin), a Russian logician of the ultra-finitist school who was imprisoned in a mental institution in Soviet Russia. Yessenin-Volpin was once asked how far one can take the geometric series of powers of 2, say (2^1, 2^2, 2^3, …, 2^{100}). He replied that the question "should be made more specific." He was then asked if he considered 2^1 to be "real," and he immediately answered yes. He was then asked if 2^2 was "real." Again he replied yes, but with a barely perceptible delay. Then he was asked about 2^3, and yes, but with more delay. These questions continued until it became clear how Yessenin-Volpin was going to handle them. He would always answer yes, but he would take 2^{100} times as long to answer yes to 2^{100} than he would to answering to 2^1. YesseninVolpin had developed his own way of handling a paradox of infinity."

(Graham & Kantor: *Naming Infinity*, 2009, 23)
(http://www.math.osu.edu/~friedman.8/pdf/Princeton532.pdf)

Ludwig Wittgenstein

Bemerkungen über die Grundlagen der Mathematik

- "feasibility" core concept
- ciritical remarks about infinity

- But was Wittgenstein a strict finitist?
- Not necessarily the best "compagnon de route"
- No worked out proposal

Various interpretations:

- Charles Kielkopf: *Strict finitism. An examination of Ludwig Wittgenstein's remarks on the foundations of mathematics* (1970)
- Crispin Wright: *Wittgenstein on the Foundations of Mathematics* (1980)
- Mathieu Marion: *Wittgenstein, Finitism and the Foundations of Mathematics* (1998)
- Victor Rodych: "Wittgenstein's Anti-Modal Finitism," *Logique et Analyse* (2000)

Some fragments:

"II-58: "Ought the word 'infinite' to be avoided in mathematics?" Yes; where it appears to confer a meaning upon the calculus, instead of getting one from it."

"V-9(e): However queer it sounds, the further expansion of an irrational number is a further expansion of mathematics."

"V-21(e): Then is infinity not actual - can I not say: "these two edges of the slab meet at infinity"? Say, not: "the circle has this property because it passes through the two points at infinity ..."; but: "the properties of the circles can be regarded in this (extraordinary) perspective". It is essentially a perspective, and a far-fetched one."

David Van Dantzig

- 1956: "Is 10^10^10 a finite number?"
- no explicit proposal
- clearly formulated defense

- different finite numeral systems
  - natural numbers one can write down
  - sums of these numbers
  - products of the two sorts of numbers above
  - …

Similar proposals to be found in:

Brian ROTMAN: *Ad Infinitum ... The Ghost in Turing's Machine*. Stanford University Press, Stanford, 1993.
- "Non-Euclidean Arithmetic"
- Not a strict finitist (direct conversation)
- No longer involved with mathematics as such

David ISLES: "What Evidence is There That 2^{65536} is a Natural Number?" (NDJFL, 33(4), 1992, pp. 465-480)

However, there are precursors:

Harvard 1940-41: Alfred Tarski, W.V.O Quine and Rudolf Carnap discuss finitist mathematics

Paolo Mancosu: "Harvard 1940–1941: Tarski, Carnap and Quine on a finitistic language of mathematics for science". *History and Philosophy of Logic*, 26(4), 2005.

"He allows a function symbol ' for successor and thus the (potential) generation of infinitely many terms (from 0). If the last thing is denoted by 'k', the problem is to give an interpretation for k', k", etc. He looks at three possibilities:
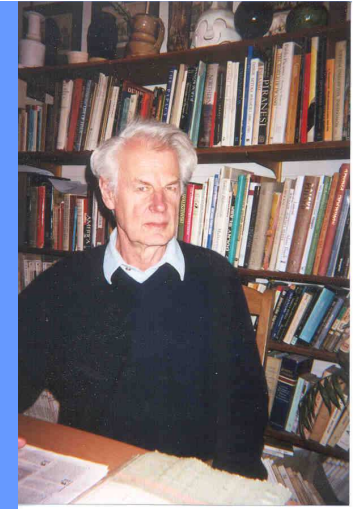
(a) k' = k" = ... = k

(b) k' = k" = ... = 0

(c) k' = 0; k" = 0', etc.

In the end he chooses (a) and modifies the axioms of arithmetic so that they are compatible with (a) (for instance " if x' = y' then x = y " needs to be dropped)."

Jan Mycielski (1989)

Both infinite and finite models:
- (s1) $\sim(\exists n)(s(n) = 0)$
- (s2) $(\forall n)(\forall m)((s(n) = s(m) \supset (n = m))$

- (s2)* $(\forall n)(\forall m)((n \neq s(n) \text{ \& } m \neq s(m)) \supset$
  $$(s(n) = s(m) \supset (n = m)).$$

(s2)* is perfectly compatible with
- (s3) $(\exists n)(n = s(n))$

"Closer to home":

Graham Priest

- paraconsistent logic
- not a strict finitist

Jean Paul Van Bendegem

- just the finite models
- both arithmetic and geometry

Graham PRIEST: "'What Could the Least Inconsistent Number be?". *Logique et Analyse*, 1994.

Graham PRIEST: "Inconsistent Models of Arithmetic I: Finite Models". *Journal of Philosophical Logic*, 1997

Graham PRIEST: "Inconsistent Models of Arithmetic II: the General Case", *Journal of Symbolic Logic*, 2000.

Jean Paul VAN BENDEGEM: "Strict Finitism as a Viable Alternative in the Foundations of Mathematics". *Logique et Analyse*, 1994.

Jean Paul VAN BENDEGEM: "Classical Arithmetic is Quite Unnatural". *Logic and Logical Philosophy*, 2003.

Jean Paul VAN BENDEGEM: "Finitism in Geometry". *The Stanford Encyclopedia of Philosophy,* 2009.

Let PA be the theory of (Peano) arithmetic. The language of PA consists of:

- the language of first-order predicate logic in its standard form
- with predicates restricted to "=" (equality) and
- functions restricted to "S" (successor), "+" (addition) and "." (multiplication). "O" is the only constant of the language.

A model M of PA is a triple M = <N, I, v_I> where N is the (standard) domain (of the natural numbers), I is an interpretation function and v_I is a valuation function based on I, satisfying the following conditions:

(I1) I(O) = 0 (where 0 is the number zero in the domain)

(I2) I(Sx) = I(x) $\oplus$ 1 (where $\oplus$ means addition in the model)

(I3) I(x + y) = I(x) $\oplus$ I(y)

(I4) I(x.y) = I(x) $\otimes$ I(y) (where $\otimes$ means multiplication in the model)

(I5) I(=) = {<n,n> | n $\in$ N}

(V1) $v_I(x = y) = 1$ iff $\langle I(x), I(y) \rangle \in I(=)$ (or, equivalently, $I(x) = I(y)$)

(V2) $v_I(\sim A) = 1$ iff $v_I(A) = 0$

(V3) $v_I(A \vee B) = 1$ iff $v_I(A) = 1$ or $v_I(B) = 1$

(V4) $v_I((\exists x)A(x)) = 1$ iff there is an I' that differs from I at most in the value of $I(x)$ such that $v_{I'}(A(x)) = 1$.

(alternatives are possible for (V4))

A formula A is valid ($\vDash_{cl} A$) iff for all models M, $v_I(A) = 1$.

Consider N* = {[0], [1], [2], ..., [L, L+1, ...]}.

Unless otherwise indicated, L is considered to be a fixed number. Read [n] as an equivalence class under a (non-stipulated) equivalence relation, or as a partition of N in a finite set of parts.

Change the (classical) logic for (the predicate extension of) LP

This will lead to an accompanying model for any classical model

A few words on LP (Logic of Paradox)

Developed by F. G. Asenjo and Graham Priest
Semantically two changes are made:

• Instead of {0, 1}, use the set of truth-values
      {{0}, {1}, {0,1}}

• Split the semantical conditions in two:
      Instead of   v(~ A) = 1   iff   v(A) = 0
      Write   1 ∈ v(~ A) iff   0 ∈ v(A)
                    0 ∈ v(~ A) iff   1 ∈ v(A)

Given N* = {[0], [1], [2], ..., [L, L+1, ...]}

(I1*) I*(O) = [0]
(I2*) I*(Sx) = [I(Sx)]
(I3*) I*(x + y) = [I(x) ⊕ I(y)]
(I4*) I*(x.y) = [I(x) ⊗ I(y)]

Given any interpretation function I and any predicate P, the positive and negative extension, resp. I+(P) en I-(P) can be defined

(I5*)   <I*(x), I*(y)> ∈ I*+(=) iff there is a n ∈ [I(x)], and there is a m ∈ [I(y)], such that
            <n, m> ∈ I+(=)
        <I*(x), I*(y)> ∈ I*-(=) iff there is a n ∈ [I(x)], and there is a m ∈ [I(y)], such that
            <n, m> ∈ I-(=)

(V1*) $1 \in v^*_{I^*}(x = y)$ iff $\langle I^*(x), I^*(y)\rangle \in I^*+(=)$
       $0 \in v^*_{I^*}(x = y)$ iff $\langle I^*(x), I^*(y)\rangle \in I^*-(=)$

(V2*) $1 \in v^*_{I^*}(\sim A)$ iff $0 \in v^*_{I^*}(A)$
       $0 \in v^*_{I^*}(\sim A)$ iff $1 \in v^*_{I^*}(A)$

(V3*) $1 \in v^*_{I^*}(A \vee B)$ iff $1 \in v^*_{I^*}(A)$ or $1 \in v^*_{I^*}(B)$
       $0 \in v^*_{I^*}(A \vee B)$ iff $0 \in v^*_{I^*}(A)$ and $0 \in v^*_{I^*}(B)$

(V4*) $1 \in v^*_{I^*}((\exists x)A(x))$ iff there is an I*' that differs from
       I* at most in the value of I*(x) such that
              $1 \in v^*_{I^{*\prime}}(A(x))$.
       $0 \in v^*_{I^*}((\exists x)A(x))$ iff for all I*' that differ from I* at
       most in the value of I*(x), $0 \in v^*_{I^{*\prime}}(A(x))$.

A formula A is r*-valid ($\vDash_{r^*} A$) iff $1 \in v^*_{I^*}(A)$ for all models M*.

Lemma: For any formula A, for any classical model M and its derived model M*, we have:

    (a) If $v_l(A) = 1$, then $1 \in v^*_{l*}(A)$,
    (b) If $v_l(A) = 0$, then $0 \in v^*_{l*}(A)$.

Theorem: if $\vDash_{cl} B$ then $\vDash_{r*} B$.

Example: $(\exists x)(x > L)$
     True if x in [L, L+1, ...] is, e.g., L+1
     False if x in [L, L+1, ...] is, e.g., L itself
Thus: $v^*_{l*}((\exists x)(x > L)) = \{0,1\}$

Leads to the idea of a more general scheme

(i) Take any first-order theory T with a finite number of predicates. Let M be a model of T.

(ii) Reformulate T in a paraconsistent fashion, extending the truth values to {{0}, {0,1}, {1}} instead of {0,1}.

(iii) If the models of M are infinite, define an equivalence relation R over the domain D of M, such that D/R is finite. (Or, equivalently, define a partition in a finite set of parts of the domain D of M. Let the resulting model be M/R or M*.)

(iv) The model M/R or M* is a finite paraconsistent model of the given first-order theory T such that validity is extended

Thus M/R is a strict and rich finitist extension of M.

Possible extensions

$Z^* = \{[\ldots, -(L+1), -L], \ldots, [-2], [-1], [0], [1], [2], \ldots,$
$[L, L+1, \ldots]\}$

$Q^* =$
(a) $Q_L = [x \mid x \in Q \,\&\, x \geq L]$, $Q_{-L} = [x \mid x \in Q \,\&\, x \leq -L]$
(b) $Q_d = \{ [p/q] \mid p/q \in Q \,\&\, -L < p,q < L \,\&\, q \neq 0\}$
(c) $Q_{ind} = \{]x, y[\mid x, y \in Q \,\&\, (x, y \in Q_d$ or $x = -L$ or
$y = L) \,\&\, \sim(Ez)(x < z < y \,\&\, z \in Q_d)\}$.

Nice consequence: $v^*_{I^*}(7 \times 1/7) = \{1\}$

Slightly growing interest in strict finitism:

Manuel BREMER: "Varieties of Finitism."
Metaphysica, 2007.

Jim MAWBY: *Strict Finitism as a Foundation of Mathematics*. Glasgow: Ph.D., 2005.

Avril STYRMAN: *Finitist Critique on Transfinity: An Investigation of Infinity, Collection Theory and Continuum*. Helsinki: Master thesis, 2009.

Feng YE: *Strict Finitism and the Logic of Mathematical Applications*, New York: Springer, 2011.

All that preceeded is the "core" of strict finitism.

There is a "belt" around this core that contains heaps of material (but not always useful):

1. Classical mathematical theories extended with a "feasibility" operator

Examples:

For arithmetic: Rohit Parikh: "Existence and Feasibility in Arithmetic". *Journal of Symbolic Logic*, 1971.

For set theory: Shaughan Lavine: *Understanding the Infinite*. Harvard UP, 1994.

2. Weak subsystems of arithmetic

Examples:

Presburger arithmetic: only successor and addition.

M. PRESBURGER: "Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt." Warschau: *Comptes Rendus du I congrés de Mathématiciens des Pays Slaves*, 1929.

Robinson arithmetic (or the system Q): axiom of induction weakened or left out

Edward NELSON: *Predicative Arithmetic.* Princeton UP, 1986.

3a. Geometry

J. T. Hjelmslev. *Die Natürliche Geometrie*, Hamburg: Gremmer & Kröger, 1923.

G. Järnefelt: "Reflections on a Finite Approximation to Euclidean Geometry: Physical and Astronomical Prospects", *Annales Academiae Scientiarum Fennicae,* 96, 1951.

P. Kustaanheimo: "A Note on a Finite Approximation of the Euclidean Plane Geometry". *Societas Scientiarum Fennica,* 1951.

Based on the idea of a finite field to replace the field of real numbers.

3b. Geometry with physical inspiration

L. Silberstein. *Discrete Spacetime. A Course of Five Lectures delivered in the McLennan Laboratory*, Toronto: University of Toronto Press, 1936.

E. Biser: "Discrete Real Space", *Journal of Philosophy*, 1941.

H.R. Coish: "Elementary particles in a finite world geometry", *Physical Review*, 1959.

T. Regge: "General relativity without coordinates". *Nuovo Cimento*, 1961.

I. Dadić & K. Pisk: "Dynamics of Discrete-Space Structure", *International Journal of Theoretical Physics*, 1979.

M. Requardt: "Discrete Mathematics and Physics on the Planck-Scale". *e-print*: arXiv:hep-th/9504118v1, 1995.

Example from E. Biser (1941):

"Let us now recapitulate the conditions of discreteness of actual regions in order to apply them to this abstract system. They are as follows: (1) there are no points, (2) a region is divisible into a finite number of cells of voluminous extension, and (3) a cell is an irreducible quantum of extension, below which an entity can not be further subdivided without losing its actuality." (p. 522)

Very often:

- Translation of differential equations in difference equations
- Classical mathematics in the background to prove existence results
- Focused on the physical constants
- Different from author to author
- Little or no impact

As far as history is concerned:

- Rather confused story
- Input from many and very different sources
- Much of the material is not known
- Remains a minority view
- Many partial proposals around
- In most cases no philosophical motivation
- Vague intuitions and ideas

There is no "poverty" involved

(a) $(\forall x)\sim xRx$

(b) $(\forall x)(\exists y)xRy$

(c) $(\forall x)(\forall y)(\forall z)((xRy \ \& \ yRz) \supset xRz)$.

All models of these three sentences are
  infinite

What does the strict finitist read?

(imagine a model with 3 elements, whereof 2 labeled)

(a)* $(\forall x)(((x = a) \lor (x = b)) \supset {\sim}xRx)$

(b)* $(\forall x)(\exists y)(((x = a) \lor (x = b))\ \&\ (y = a) \lor$
$(y = b)) \supset xRy)$

(c)* $(\forall x)(\forall y)(\forall z)((((x = a) \lor (x = b))\ \&$
$((y = a) \lor (y = b))\ \&\ ((z = a) \lor (z = b))) \supset$
$((xRy\ \&\ yRz) \supset xRz))$

Note: model corresponds to {a, b, *} as partition of
the infinite domain. * contains all other elements

The "accountant's" model:

Easy to prove that
$$(\forall x)(\forall y)(xRy \supset {\sim}yRx)$$

Pick a and b, such that aRb and suppose that bRa is the case, then by transitivity we have aRa which contradicts irreflexivity (all this under the assumption that we can talk about a and b)

Inspired by Tarski's remark:

"Moreover, one should allow from the start variables in the language which range over all things in the world; but it remains open how many things there are.

Instead of saying 'prod(2, 3)=6', he proposes: 'whenever x is a successor of a successor of 0, and y . . ., and z . . ., then prod(x, y)=z'."
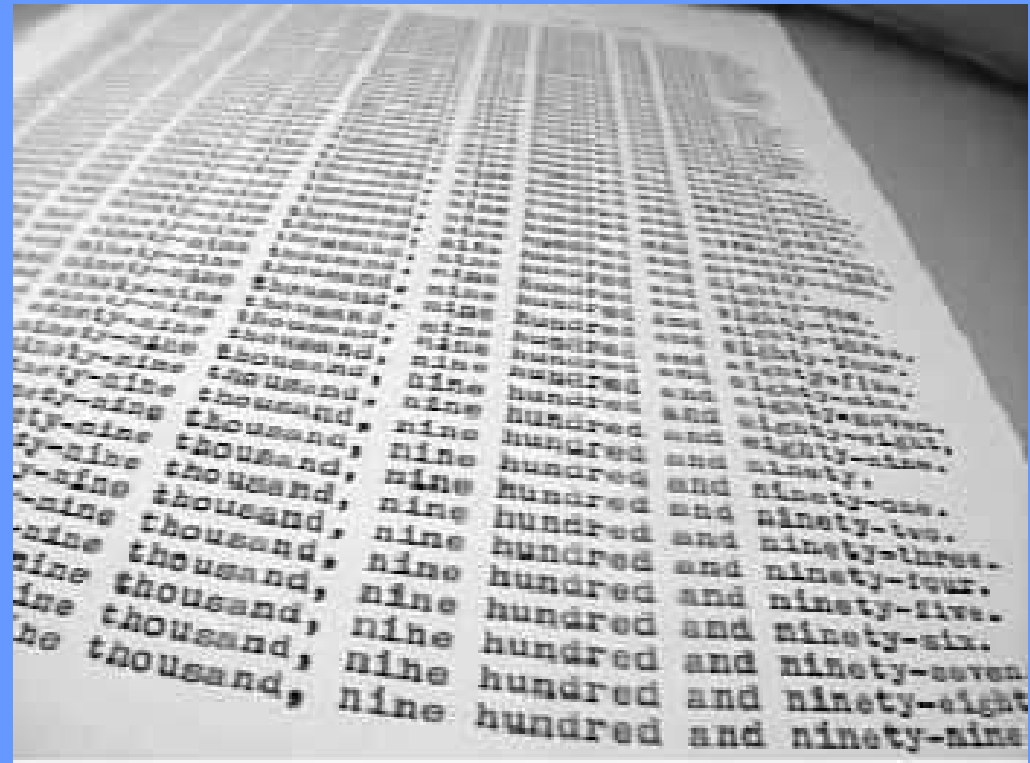
(Mancosu, 2005, p. 351)

Compare to mathematical practice

On the one hand: proof that prime numbers are not bounded by a finite number

On the other hand: $2^{57,885,161} - 1$ is the largest prime number known at the present

These two statements are perfectly compatible

Les Stewart (Australia) spent 15 years from 1983 to 1998 typing out numbers from one to one million in letters

Whatever he did, it was not mathematics!