APPLICATIONS OF ELEMENTARY SUBMODELS IN GENERAL TOPOLOGY

STEFAN GESCHKE

Several theorems in general topology, especially inequalities between certain cardinal invariants of a topological space, can be proved in the following way:

For a given topological space (X, τ) consider a space (X_0, τ_0) which is small in some sense and approximates (X, τ) sufficiently well. Calculate the cardinal invariant in question for (X_0, τ_0) and show that this cardinal invariant is the same for (X_0, τ_0) and (X, τ) since (X_0, τ_0) is a good approximation of (X, τ).

The method of elementary submodels provides a uniform approach for generating small approximations of topological spaces as well as of other structures. Typically, mathematics takes place in the set theoretic universe $(V, \in)$. For a given topological space (X, τ) we would like to use the Löwenheim-Skolem Theorem to get a small elementary submodel $(M, \in \cap M^2)$ of the universe s.t. $(X, \tau) \in M$ and then consider the space (X_0, τ_0) which is what M thinks that (X, τ) is. However, there are two problems. First of all, working in V itself we cannot get elementary submodels of $(V, \in)$. This follows from Gödel's Second Incompleteness Theorem. Second, it is not clear what it really is that *M thinks that (X, τ) is.*

The first problem can be solved by taking elementary submodels not of the whole universe but of a sufficiently large initial segment of the universe. The solution of the second problem depends on the application we have in mind.

To illustrate the method of elementary submodels we give a proof of Arkhangel'skii's theorem that the weight of a continuous image of a compact space is not larger than the weight of the space itself using this technique.

References

[1] H. Ebbinghaus, J. Flum, W. Thomas, *Einführung in die Mathematische Logik*, B.I. Verlag (1992)

[2] R. Engelking, *General Topology*, Polska Akademie Nauk, Monographie Matematyczne, Vol. 60, Panstowe Wydawnictwo Naukowe - Polish Scientific Publishers, Warszawa (1977)

[3] W. Just, M. Weese, *Discovering modern set theory II: Set-theoretic tools for every mathematician*, American Mathematical Society, Graduate Studies in Mathematics, Volume 18 (1997)

II. Math. Institut der FU-Berlin, Arnimallee 3, 14195 Berlin