Kunen, Set Theory (1980)

pp. 216-217

6.18. THEOREM. *If ZFC is consistent, so are:*
(a) ZFC + CH + $2^{\omega_1} = \omega_2$ + $2^{\omega_2} = \omega_{\omega_8}$.
(b) ZFC + CH + $2^{\omega_1} = \omega_5$ + $2^{\omega_2} = \omega_7$.
(c) ZFC + $2^{\omega} = \omega_3$ + $2^{\omega_1} = \omega_4$ + $2^{\omega_2} = \omega_6$.

PROOF. In all cases, start with M satisfying ZFC + GCH.

For (a) Let $\mathbb{P} = (\mathrm{Fn}(\omega_{\omega_8} \times \omega_2, 2, \omega_2))^M$. By Theorem 6.17, $\mathbb{P}$ preserves cardinals and if G is $\mathbb{P}$-generic over M, $(2^{\omega_2} = \omega_{\omega_8})^{M[G]}$. The fact that $2^{\omega_1} = \omega_2$ holds in $M[G]$ follows from the fact that $({}^{(\omega_1)}2)^M = ({}^{(\omega_1)}2)^{M[G]}$ by Theorem 6.14. Thus, if $F \in M$ and (F maps ω_2 onto ${}^{(\omega_1)}2)^M$, then (F maps ω_2 onto ${}^{(\omega_1)}2)^{M[G]}$. Likewise, $(2^{\omega} = \omega_1)^{M[G]}$.

For (b) we force twice. Let $\mathbb{P} = (\mathrm{Fn}(\omega_7 \times \omega_2, 2, \omega_2))^M$, let G be $\mathbb{P}$-generic over M, and let $N = M[G]$; then as in (a),

$$(2^{\omega} = \omega_1 \wedge 2^{\omega_1} = \omega_2 \wedge 2^{\omega_2} = \omega_7)^N.$$

Furthermore, $(\kappa^{\omega_1} = \kappa)^N$ whenever $(\kappa \geq \omega_2 \wedge \kappa$ is regular$)^N$, since this is true in M by $(\mathrm{GCH})^M$, and $({}^{(\omega_1)}\kappa)^M = ({}^{(\omega_1)}\kappa)^N$. We now apply our results on forcing with N as the ground model instead of M. Let

$$\mathbb{Q} = (\mathrm{Fn}(\omega_5 \times \omega_1, 2, \omega_1))^N.$$

By $(2^{<\omega_1} = \omega_1)^N$, $\mathbb{Q}$ preserves cardinals. Let H be $\mathbb{Q}$-generic over N. $(\mathrm{CH})^{N[H]}$ is proved as in (a). $(2^{\omega_2} \geq \omega_7)^{N[H]}$ follows from $(2^{\omega_2} \geq \omega_7)^N$. To see that in fact $(2^{\omega_2} = \omega_7)^{N[H]}$, use the method of Theorem 6.17; namely, in N, $\mathbb{Q}$ has the ω_2-c.c. and $|\mathbb{Q}| = \omega_5^{\omega_1} = \omega_5$, so there are only $((\omega_5)^{\omega_1})^{\omega_2} = \omega_7$ nice names for subsets of ω_2. To see that $(2^{\omega_1} = \omega_5)^{N[H]}$, apply Theorem 6.17 directly, plus the fact that $(\omega_5^{\omega_1} = \omega_5)^N$.

For (c), force three times, and construct $M \subset N_1 \subset N_2 \subset N_3$. N_1 satisfies $2^{\omega} = \omega_1 \wedge 2^{\omega_1} = \omega_2 \wedge 2^{\omega_2} = \omega_6$, N_2 satisfies $2^{\omega} = \omega_1 \wedge 2^{\omega_1} = \omega_4 \wedge 2^{\omega_2} = \omega_6$, and N_3 satisfies (c). □

In proving (b) and (c), it is very important that we proceed *backwards*, dealing with the largest cardinal first. For example, if we tried to prove (b) by letting $\mathbb{P} = (\mathrm{Fn}(\omega_5 \times \omega_1, 2, \omega_1))^M$ and $N = M[G]$, where G is $\mathbb{P}$-generic over M, then N would satisfy $2^{\omega_1} = \omega_5$. Thus, $(2^{<\omega_2} \neq \omega_2)^N$, so if

we set $\mathbb{Q} = (\mathrm{Fn}(\omega_7 \times \omega_2, 2, \omega_2))^N$, $\mathbb{Q}$ would not preserve cardinals. In fact, if H is $\mathbb{Q}$-generic over N, then $(\omega_5)^N$ would have cardinality ω_2 in $N[H]$ (see Exercise G3), and $N[H]$ would satisfy $2^{\omega_1} = \omega_2$.