SET THEORY
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2 BASIC PROBLEMS OF MATH SOLVED

Proof Involves Set Theory, Now Used in Schools

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Two of the most fundamental questions in mathematics today have been answered by Paul J. Cohen, a young Stanford University mathematician.

The two questions had persisted for more than a quarter century. By answering them, Dr. Cohen demonstrated the power of modern mathematics. Ironically, he exposed some of its weakness as well.

M[G] SATISFIES ZFC !

1. A PROOF USES ONLY FINITELY MANY AXIOMS $\varphi_1, \ldots, \varphi_k$
IT SUFFICES TO CONSTRUCT AN INTERPRETATION OF $(\in, =)$ THAT SATISFIES $\varphi_1 \ldots, \varphi_k$ BUT NOT φ

φ:
- $\neg CH$
- CH
- SUSLIN'S HYP.

2. VIA REFLECTION, AND LÖWENHEIM–SKOLEM, AND MOSTOWSKI
WE FOUND A COUNTABLE TRANSITIVE M THAT SATISFIES $\varphi_1, \ldots, \varphi_k$.

3. WE SIMPLY (LAZILY) ASSUME M SATISFIES ALL OF ZFC.

4. WE CAN EXTEND SUCH M IN A SYSTEMATIC WAY

5. - $\mathbb{P}$ A PARTIAL ORDER (WITH MAXIMUM $\mathbb{1}$) IN M
- G AN M-GENERIC FILTER ON $\mathbb{P}$
  - FILTER:
    - $p, q \in G \rightarrow \exists r \in G \ \ r \leq p, q$
    - $p \in G \wedge p \leq q \rightarrow q \in G$
  - M-GENERIC: IF $D \in M$ IS DENSE IN $\mathbb{P}$ THEN $G \cap D \neq \emptyset$

    $\forall p \, \exists d \in D \ \ d \leq p$
  - COUNTABILITY OF M: SUCH G EXIST. EVERY $p \in \mathbb{P}$ SITS IN A G.
    ($\mathbb{P}$-GENERIC ON M)

USEFUL NOTIONS

$p \perp q$ NO r WITH $r \leq p, q$
(INCOMPATIBLE)

p AND q CAN NEVER BE
IN THE SAME GENERIC FILTER G.

COMPATIBLE $\exists r \; r \leq p, q$

ANTICHAIN: A $p \neq q$ IN A
$\Rightarrow p \perp q$

ZORN'S LEMMA: MAXIMAL
ANTICHAINS EXIST

G IS M-GENERIC IFF
G INTERSECTS EVERY MAX. ANTICH
(IN ONE POINT)

- A MAX ANTICH IN D IS MAXIMAL IN $\mathbb{P}$
- A MAX AC: $D = \{q : (\exists p \in A)(q \leq p)\}$ IS DENSE

$M^{\mathbb{P}}$: THE CLASS OF $\mathbb{P}$-NAMES

$\tau \in M^{\mathbb{P}}$: τ IS A RELATION
$\operatorname{DOM} \tau \subseteq M^{\mathbb{P}}$
$\operatorname{RAN} \tau \subseteq \mathbb{P}$

τ_G

$\operatorname{VAL}(\tau, G) = \{\operatorname{VAL}(\pi, G) : (\exists p \in G)(\langle \pi, p \rangle \in \tau)\}$

$M[G] = \{\operatorname{VAL}(\tau, G) : \tau \in M^{\mathbb{P}}\}$

TRUTH IN $M[G]$ VERSUS
TRUTH IN M

WE DEFINED

$p \Vdash \varphi(\tau_1, \ldots, \tau_k)$ IFF FOR ALL G
WITH $p \in G$
$\varphi(\tau_{1,G}, \ldots, \tau_{k,G})^{M[G]}$

~~FORCES~~ \VDASH

IF $\mathbb{P}$ IS NON-TRIVIAL $\forall p \, \exists q, r \leq p \;\; q \perp r$

THEN NO G IS IN M

$\mathbb{P} \setminus G$ IS DENSE

WE ALSO DEFINED (USING ONLY STUFF FROM M)

$p \Vdash^* \varphi(\tau_1, \ldots, \tau_k)$ USING NAMES AND $\mathbb{P}$

① $\tau_1 = \tau_2$ ② $\tau_1 \in \tau_2$

③ $\varphi \wedge \psi$, $\neg \varphi$

④ $\exists \sigma\, \varphi(\sigma, \tau_1, \ldots, \tau_k)$ $\exists \sigma \in M$

TECHNICAL RESULT:

① IF $p \in G$ AND $(p \Vdash^* \varphi(\tau_1, \ldots, \tau_k))^M$

THEN $(\varphi(\tau_{1G}, \ldots, \tau_{kG}))^{M[G]}$

② IF $(\varphi(\tau_{1G}, \ldots, \tau_{kG}))^{M[G]}$

THEN $(\exists p \in G)(p \Vdash^* \varphi(\tau_1, \ldots, \tau_k))^M$

$p \Vdash^* \tau_1 = \tau_2$

FAILS

(i) FOR ALL $\langle \pi_1, s_1 \rangle \in \tau_1$

$D(\pi_1, s_1, p) =$ $\neg(A \to B) \equiv A \wedge \neg B$

$\{q \leq p : q \leq s_1 \to (\exists \langle \pi_2, s_2 \rangle \in \tau_2)(q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$

IS DENSE BELOW p

(ii) —— SYMMETRIC ——

① IF $(p \Vdash^* \tau_1 = \tau_2)^M$ $s_1 \in G$

TAKE $x \in \tau_{1G}$ ---- $x \in \tau_{2G}$

$x = \pi_{1G}$ WHERE $\langle \pi_1, s_1 \rangle \in \tau_1$

TAKE $r \in G$ WITH $r \leq p, s_1$

$r \Vdash^* \tau_1 = \tau_2$

THERE IS $q \leq r$ IN $D(\pi_1, s_1, r) \cap G$

$q \leq r \leq s_1$

HENCE THERE IS $\langle \pi_2, s_2 \rangle \in \tau_2$

WITH $q \leq s_2$ AND $q \Vdash^* \pi_1 = \pi_2$

$s_2 \in G$: $\pi_{2,G} \in \tau_{2,G}$

$q \Vdash^* \pi_1 = \pi_2$

INDUCTION $\pi_{1G} = \pi_{2G}$ SO $x \in \tau_{2G}$.

(2) IF $\tau_{1G} = \tau_{2G}$
THEN $(\exists p \in G)(p \Vdash^* \tau_1 = \tau_2)^M$

LOOK FOR A DENSE SET

- $D_= \{ r : r \Vdash^* \tau_1 = \tau_2 \}$ ⟵ G MUST INTERSECT THIS
- $D_i = \{ r : (\exists \langle \pi_1, s_1 \rangle \in \tau_1)(r \leq s_1 \wedge$
$(\forall \langle \pi_2, s_2 \rangle \in \tau)(\forall q \in \mathbb{P})(\underline{q \leq s_2} \wedge \cancel{q \Vdash^* \pi_1 = \pi_2}) \to \cancel{q \perp r}) \}$
- D_{ii} — — SYMMETRIC — —

- $(D_=) \cup D_i \cup D_{ii}$ IS DENSE
- $G \cap D_i = G \cap D_{ii} = \emptyset$

- $r \in G \cap D_i$ WE GET $\langle \pi_1, s_1 \rangle \in \tau_1$ WITH $r \leq s_1$
  SO $s_1 \in G$ AND $\pi_{1G} \in \tau_{1G}$
  IF $x \in \tau_{2G}$ THEN $x = \pi_{2G}$ WITH $s_2 \in G$
  AND $\langle \pi_2, s_2 \rangle \in \tau_2$

  IF $\pi_{1G} = \pi_{2G}$ THEN BY INDUCTION
  FIND $p \in G$ WITH $p \Vdash^* \pi_1 = \pi_2$
  NOW TAKE $q \in G$, $q \leq p, r, s_2, s_1$
  $q \leq s_2$ AND $q \| r$ SO $\cancel{q \Vdash^* \pi_1 = \pi_2}$
  $q \leq p$ SO $q \Vdash^* \pi_1 = \pi_2$

DITTO $G \cap D_{ii} = \emptyset$

TAKE p SUCH THAT $\cancel{p \Vdash^*} \tau_1 = \tau_2$

SAY (i) FAILS

WE GET $\langle \pi_1, s_1 \rangle$ WHERE
  $D(\pi_1, s_1, p)$ IS NOT DENSE BELOW p
  $r \leq p$ S.T. NO $q \leq r$ IS IN $D(\pi_1, s_1, p)$
  $\forall q \leq r \, (q \leq s_1 \wedge (\forall \langle \pi_2, s_2 \rangle \in \tau_2) \neg (q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2))$
  (ALSO $r \leq s_1$)
  IF $\langle \pi_2, s_2 \rangle \in \tau_2$ AND $q \leq s_2$ AND $q \Vdash^* \pi_1 = \pi_2$
  THEN $q \perp r$
  IF $s \leq q, r$ THEN
  $s \leq s_2 \wedge s \Vdash^* \pi_1 = \pi_2$ $(s \leq q)$
  AND $\neg(s \leq s_2 \wedge s \Vdash^* \pi_1 = \pi_2)$ $(s \leq r)$
  CONTRADICTION
  SO $r \in D_i$

IN PRACTICE

① $p \Vdash \varphi(\tau_1, \ldots, \tau_k)$ IFF $(p \Vdash^{*}_{\varphi} \varphi(\tau_1, \ldots, \tau_k))^M$

② $\varphi(\tau_{1G}, \ldots, \tau_{k,G})^{M[G]}$ IFF $(\exists p \in G)(\varphi(\tau_1, \ldots, \tau_k))$

① DEFINABILITY OF $\Vdash$
② TRUTH LEMMA

ZFC TRANSITIVITY: EXTENSIONALITY
REGULARITY/FOUNDATION
PAIRING, UNION: EASY NAMES
INFINITY : $\omega \in M[G]$
POWERSET : EASY NAME (RELATIVELY)
AC : LET $\sigma \in M^{\mathbb{P}}$

$\langle \pi_\gamma : \gamma \in \alpha \rangle$ ENUMERATES $\mathrm{DOM}\, \sigma$

$$\tau = \{ op(\check{\gamma}, \pi_\gamma) : \gamma \in \alpha \} \times \{ \mathbb{1} \}$$

$$\tau_G = \{ \langle \gamma, \pi_{\gamma, G} \rangle : \gamma \in \alpha \}$$

$$\mathrm{DOM}\, \tau_G = \alpha$$

$$\sigma_G \subseteq \mathrm{RAN}\, \tau_G.$$

— SEPARATION

GIVEN $\sigma, \tau_1, \ldots, \tau_k$ (τ) AND $\varphi(x, v, y_1, \ldots, y_k)$

WE WANT $\{ a \in \sigma_G : \varphi(a, \sigma_G, \tau_G)^{M[G]} \} \in M[G]$

$$\varsigma = \{ \langle \pi, p \rangle \in \mathrm{DOM}(\sigma) \times \mathbb{P} : p \Vdash (\pi \in \sigma \wedge \varphi(\pi, \sigma, \tau)) \}$$

THEN ς_G IS AS REQUIRED

— REPLACEMENT.

IF $\forall x \exists! y\ \varphi(x, y, s, t_1, \ldots, t_k)$

THEN $\forall A \exists B\ \forall x \in A\ \exists y \in B\ \varphi(x, y, A, t_1, \ldots, t_k)$

GIVEN $\sigma, \tau_1, \ldots, \tau_k$

WE WANT $\varsigma \in M^{\mathbb{P}}$

$$(\forall x \in \sigma_G\ \exists y \in \varsigma_G\ \varphi(x, y, \sigma_G, \tau_{1G}, \ldots, \tau_{kG}))^{M[G]}$$

IN M USE REFLECTION: THERE IS A β SUCH THAT

$$\forall \pi \in \mathrm{DOM}\, \sigma\ \forall p \in \mathbb{P} [\exists \mu \in M^{\mathbb{P}}\ p \Vdash \varphi(\pi, \mu, \sigma, \tau) \Longrightarrow \exists \mu \in M^{\mathbb{P}} \cap V_\beta\ p \Vdash \varphi(\pi, \mu, \sigma, \tau)]$$

TAKE $R = (V_\rho \cap M^{\mathbb{P}})$

AND $\mathcal{S} = R \times \{\mathbb{1}\}$

THIS NAME WORKS.

BACK TO $\neg$CH

COHEN STARTED WITH GCH^M.

GÖDEL'S CONSTRUCTIBLE UNIVERSE L

$L \models GCH + AC$.

- $\mathbb{P} = Fn(\omega_2^M \times \omega_1 2)$
- G GENERIC : $\bigcup G : \omega_2^M \times \omega \to 2$

$f(\alpha) = \{ n : \bigcup G(\alpha, n) = 1 \}$

$f : \omega_2^M \longrightarrow \mathcal{P}^{M[G]}(\omega)$ INJECTIVE

- GENERAL LEMMA

IF $X, Y \in M$ AND $g : X \to Y$ IN $M[G]$

THEN THERE IS $F : X \longrightarrow [Y]^{\leq \aleph_0}$ IN M

SUCH THAT $(\forall x \in X)(g(x) \in F(x))$

"EVERY MAP IN $M[G]$ IS CAPTURED BY A COUNTABLE PIPE FROM M"

TAKE $\gamma \in M^{\mathbb{P}}$ WITH $g = \gamma_G$.

AND $p \in G$ SUCH THAT

$$p \Vdash \gamma : \check{X} \longrightarrow \check{Y}$$

IN M DEFINE

$$R_\gamma = \{ \langle q, \langle x, y \rangle \rangle : q \in \mathbb{P},\ q \leq p \to q \Vdash \gamma(\check{x}) = \check{y},\ q \perp p \to y = y_0 \}$$

y_0 SOME FIXED MEMBER OF Y

- $R_\gamma[G] = g$
- $F(x) = \{ y : (\exists q \in \mathbb{P})(\langle q, \langle x, y \rangle \rangle \in R_\gamma) \}$

FOR $y \in F(x)$ TAKE $q_y \in \mathbb{P}$ SUCH THAT $\langle q_y, \langle x, y \rangle \rangle \in R_\gamma$ (AC)

$y \neq z$: $q_y \perp q_z$

$r \leq q_y, q_z$: $r \Vdash \gamma(\check{x}) = \check{y},\ \gamma(\check{x}) = \check{z}$) $z = y$

$\{q_y : y \in F(x)\}$ IS AN ANTICHAIN
ANTICHAINS IN $\mathbb{P}$ ARE COUNTABLE
SO $F(x)$ IS COUNTABLE.

COROLLARY
IF $\alpha \in On \cap M$
$(\alpha \text{ IS A CARDINAL})^M$ IFF $(\alpha \text{ IS A CARDINAL})^{M[G]}$

$\Leftarrow$ $\beta < \alpha$ NO MAP $f : \beta \to \alpha$ IN $M[G]$
IS SURJECTIVE
HENCE NO MAP IN M IS ONTO
"α IS A CARDINAL" IS DOWNWARD ABSOLUTE

$\Rightarrow$ $\alpha > \omega$.
IF $\beta < \alpha$ AND $f : \beta \to \alpha$ IS
A MAP IN $M[G]$ THEN WE
HAVE $F : \beta \to [\alpha]^{\leq \aleph_0}$ IN M
WITH $f(\gamma) \in F(\gamma)$ $(\gamma \in \beta)$
BUT THEN $f[\beta] \subseteq \bigcup_{\gamma \in \beta} F(\gamma)$
AND
$|\bigcup_{\gamma \in \beta} F(\gamma)| \leq |\beta| \cdot \aleph_0 < \alpha$
SO f IS NOT ONTO

EXERCISE $cf^M \alpha = cf^{M[G]} \alpha$ FOR ALL α

CALCULATE $2^{\aleph_0}$ IN $M[G]$.

- $2^{\aleph_0} \geq \aleph_2$
- $2^{\aleph_0} \leq \aleph_2$

ASSUME $\tau_G \subseteq \omega$
TAKE $p \in G$ $p \Vdash \tau \subseteq \check{\omega}$

$\tau' = \{ \langle \check{n}, q \rangle : \begin{array}{l} q \leq p \to q \Vdash \check{n} \in \tau \\ q \perp p \to n = 0 \end{array} \}$

$\emptyset \Vdash \tau' \subseteq \check{\omega}$ $p \Vdash \tau' = \tau$
TAKE MAP $n \longrightarrow A_n \subseteq \mathbb{P}$
A_n MAX ANTICHAIN
IN $\{q : \langle \check{n}, q \rangle \in \tau'\}$

$\tau'' = \bigcup_{n \in \omega} \{\check{n}\} \times A_n$ $\sim \bigcup_{\alpha \in \omega_1} \{\check{\alpha}\} \times A_\alpha$

$\emptyset \Vdash \tau'' \subseteq \omega$

$p \Vdash \tau'' = \tau$

EVERY SUBSET OF ω HAS A 'VERY NICE' NAME.

HOW MANY SUCH NAMES?

— $|\mathbb{P}| = \aleph_2$

— $\tau'' \subseteq \mathrm{dom}(\check{\omega}) \times \mathbb{P}$

$|\mathrm{dom}(\check{\omega}) \times \mathbb{P}| = \aleph_2$

τ'' IS COUNTABLE — CARD $\aleph_1$

— SO AT MOST $\aleph_2^{\aleph_0}$ VERY NICE NAMES $\aleph_2^{\aleph_1} = \aleph_2$

GCH: $\aleph_2^{\aleph_0} = \aleph_2$

— WE FIND THAT INDEED

$|\mathcal{P}^{M[G]}(\omega)| \leq \aleph_2^M = \aleph_2^{M[G]}$

EXERCISE $2^{\aleph_1} = \aleph_2$ IN $M[G]$

$\rightarrow$ $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ IN $M[G]$.

EXCELLENT
SAME ORDINALS
SAME COF. FUNCTION.

Cohen.) Let $\mathfrak{M}$ be a ZF*-model in which $V = L$ is valid, fixed once for all.

Theorem 1. *Let $\aleph$ be an infinite cardinal of $\mathfrak{M}$ with $\aleph_0 < \mathrm{cf}(\aleph)$. Then there is an excellent extension $\mathfrak{N}$ of $\mathfrak{M}$ in which $2^{\aleph_0} = \aleph$.*

Theorem 2. *Let $\aleph$ and $\aleph'$ be infinite cardinals of $\mathfrak{M}$ with $\aleph = \mathrm{cf}(\aleph) < \mathrm{cf}(\aleph')$. Then there is an excellent extension $\mathfrak{N}$ of $\mathfrak{M}$ in which:*

(i) $2^{\aleph} = \aleph'$;

(ii) if $\aleph_\alpha < \aleph$, then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Theorem 3. *Identify the ordinary integers with an initial segment of the integers of $\mathfrak{M}$. Let $k, n_0, \ldots, n_k$ be ordinary integers and suppose that $i < n_i$ (for $0 \leqslant i \leqslant k$) and $n_0 \leqslant n_1 \leqslant \ldots \leqslant n_k$. Then there is an excellent extension $\mathfrak{N}$ of $\mathfrak{M}$ in which $2^{\aleph_i} = \aleph_{n_i}$ (for $0 \leqslant i \leqslant k$).*

"$2^{\aleph_0}$ CAN BE ANYTHING IT OUGHT TO BE"
SOLOVAY 1965

THM 1 JUST USE $\mathrm{FN}(\aleph \times \omega, 2)$

THM 2 TAKE $\aleph_1$ AND SOME $\kappa > \aleph_1$
$\mathrm{cf}\,\kappa > \aleph_1$

MAKE $2^{\aleph_1} = \kappa$
$2^{\aleph_0} = \aleph_1$

M SATISFIES $V=L$ HENCE GCH

HOMEWORK 12

$$\mathbb{P} = \mathrm{Fn}(\underline{\kappa \times \omega_1}, 2, \aleph_1)$$

G GENERIC:

$$\bigcup G : \kappa \times \omega_1 \to 2$$

$$x_\alpha = \{\eta : \bigcup G(\alpha, \eta) = 1\}$$

$\alpha \mapsto x_\alpha$ FROM κ TO $\mathcal{P}^{M[G]}(\omega_1)$

ALL ANTICHAINS IN $\mathbb{P}$ HAVE CARDINALITY AT MOST $\aleph_1$

NEW MAPS ARE CAPTURED BY PIPES THAT ARE $\aleph_1$ WIDE

IF $\alpha > \aleph_1$ IS A CARD. IN M THEN IT IS STILL A CARD IN $M[G]$.

WHAT ABOUT $\aleph_1$?

NEW PHENOMENON!!

$g : \omega \to X$ $\quad X \in M$

$g \in M[G]$

$g = \gamma_G$ WLOG $\emptyset \Vdash \gamma : \check{\omega} \to \check{X}$

TAKE $q \in \mathbb{P}$

$$q \Vdash (\exists z)(z \in \check{X} \wedge \gamma(\check{0}) = z)$$

THERE ARE $r \leq q$ AND σ

SUCH THAT $r \Vdash (\sigma \in \check{X} \wedge \gamma(\check{0}) = \sigma)$

THERE IS $s \leq r$ AND $x \in X$

ST $s \Vdash (\check{x} = \sigma \wedge \gamma(\check{0}) = \sigma)$

$$s \Vdash \gamma(\check{0}) = \check{x}$$

TAKE $q_0 \leq q$ AND $x_0 \in X$

S.T. $q_0 \Vdash \gamma(\check{0}) = \check{x}_0$

RECURSION:

$q_{n+1} \leq q_n$ AND $x_{n+1} \in X$

$$q_{n+1} \Vdash \gamma(\check{n+1}) = \check{x}_{n+1}$$

TAKE $r = \bigcup_{n\in\omega} q_n$ $\underline{\underline{\in \mathbb{P}}}$

THEN $r \leq q$

FOR ALL n $r \Vdash \dot{y}(\check{n}) = \check{x}_n$

TAKE $f = \{\langle n, x_n \rangle : n \in \omega\}$ $\underline{\underline{\in M}}$

AND $r \Vdash$ " $\dot{y} = \check{f}$ "

$D_{\dot{y}} = \{ r : (\exists f \in X^{\omega})(r \Vdash \dot{y} = \check{f}) \}$

IS DENSE IN $\mathbb{P}$

TAKE $r \in G \cap D_{\dot{y}}$ WITH ITS f

SO $y = f$ $\underline{\in M}$

1. $\mathcal{P}^M(\omega) = \mathcal{P}^{M[G]}(\omega)$

2. ω_1 STAYS A CARDINAL

$2^{\aleph_1} = \kappa$ SAME COMPUTATIONS