SET THEORY 2021-12-13

- THE ELO IS THE ULTIMATE ARBITER

- SUMMARY OF WHAT WE HAVE ACHIEVED THUS FAR.
  - TO DEFEAT A POTENTIAL PROOF OF A STATEMENT ψ, E.G., CH, ¬CH, SUSLIN'S HYPOTHESIS – TAKE FINITELY MANY (INSTANCES OF) AXIOMS SAY $\varphi_1, \dots, \varphi_n$ AND BUILD A COUNTABLE STRUCTURE N SUCH THAT $\varphi_1^N \wedge \dots \wedge \varphi_n^N \wedge (\neg\psi)^N$ HOLDS.
  - GIVEN FINITELY MANY AXIOMS $\varphi_1, \dots, \varphi_n$ WE CAN FIND A COUNTABLE TRANSITIVE M SUCH THAT $\varphi_1^M \wedge \dots \wedge \varphi_n^M$ HOLDS.
  - FOR CONVENIENCE WE ASSUME M SATISFIES ALL OF ZFC. [AFTERWARDS YOU CAN INSPECT THE PROOF AND SEE WHICH INSTANCES OF THE AXIOMS YOU USED (AND PRETEND YOU ASSUMED THAT M IS SATISFIED ONLY THOSE).
  - WE CAN EXTEND M USING A SINGLE SET G TO OBTAIN $N = M[G]$ SUCH THAT $(\neg\psi)^N$ HOLDS AND N ALSO SATISFIES ALL OF ZFC.

HOW :- TAKE A PARTIAL ORDER $\mathbb{P}$ IN M
  - G IS TO BE A FILTER ON $\mathbb{P}$
    - $p, q \in G \to (\exists r \in G)(r \leq p, q)$
    - $p \in G \wedge p \leq q \to q \in G$
  - $G \cap D \neq \emptyset$ WHENEVER $D \in M$ IS DENSE IN $\mathbb{P}$
    DENSE : $(\forall p \in \mathbb{P})(\exists q \in D)(q \leq p)$
  - SUCH G EXIST BECAUSE M IS COUNTABLE FOR EVERY $p \in \mathbb{P}$ THERE IS SUCH A G.
  - SUCH G ARE CALLED M-GENERIC FILTERS ON $\mathbb{P}$ [SOME AUTHORS: "$\mathbb{P}$-GENERIC IN M"]

USEFUL NOTIONS:
- $p \perp q$ (INCOMPATIBLE) NO r WITH $r \leq p, q$
- $p \parallel q$ (COMPATIBLE) SOME r WITH $r \leq p, q$
- ANTICHAIN $A \subseteq \mathbb{P}$: $p, q \in A \wedge p \neq q \to p \perp q$
- ZORN: MAXIMAL ANTICHAINS EXIST.

USEFUL: G IS AN M-GENERIC FILTER ON $\mathbb{P}$
IFF - G IS A FILTER
- $G \cap A \neq \emptyset$ FOR EVERY MAXIMAL ANTICHAIN IN $\mathbb{P}$ THAT IS IN M
(NOTE $|G \cap A| = 1$ IN THAT CASE)

FORCING : - BUILDING $M[G]$ OUT OF $M \cup \{G\}$
- CLOSE RELATION BETWEEN TRUTH IN $M[G]$ AND TRUTH IN M.

- $\mathbb{P}$-NAMES :
τ IS A $\mathbb{P}$-NAME
IFF τ IS A RELATION [SET OF ORDERED PAIRS]
SUCH THAT
IF $\langle \pi, p \rangle \in \tau$ THEN - π IS A $\mathbb{P}$-NAME
- $p \in \mathbb{P}$.

$M^{\mathbb{P}}$ IS THE COLLECTION OF $\mathbb{P}$-NAMES THAT ARE IN M
- FOR US IT IS A SET
- FOR M IT IS A PROPER CLASS

- $\mathrm{VAL}(\tau, G) = \{ \mathrm{VAL}(\pi, G) : (\exists p \in G)(\langle \pi, p \rangle \in \tau)\}$
ABBREVIATION τ_G (OTHER NOTATION $\tau[G]$).

- <u>WE</u> DEFINE(P) (<u>FROM THE OUTSIDE</u>):
$p \Vdash \varphi(\tau_1, \dots, \tau_k)$ IFF $\varphi(\tau_{1G}, \dots, \tau_{kG})^{M[G]}$
FOR <u>EVERY</u> M-GENERIC FILTER G ON $\mathbb{P}$ WITH $p \in G$

[NOTE: IN ALL NON-TRIVIAL CASES WE HAVE $G \notin M$
IF NON-TRIVIAL: $(\forall p \in \mathbb{P})(\exists q, r \in \mathbb{P})(q, r \leq p \wedge q \perp r)$
FOR IN THAT CASE $\mathbb{P} \setminus G$ IS DENSE]

- WE ALSO DEFINE(D) (AND RELATIVISED TO M)
$p \Vdash^* \varphi(\tau_1, \dots, \tau_k)$
BY RECURSION ON COMPLEXITY OF φ.
FIRST $\tau_1 = \tau_2$
THEN $\tau_1 \in \tau_2$
THEN $\neg\varphi$ AND $\varphi \wedge \psi$
THEN $(\exists \sigma)(\varphi(\sigma, \tau_1, \dots, \tau_k))$ ---- $(\exists \sigma \in M^{\mathbb{P}})$ ----

MAIN TECHNICAL RESULT

① IF $p \in G$ AND $(p \Vdash^* \varphi(\tau_1, \ldots, \tau_n))^M$

THEN $(\varphi(VAL(\tau_1, G), \ldots, VAL(\tau_n, G)))^{M[G]}$

② IF $(\varphi(VAL(\tau_1, G), \ldots, VAL(\tau_n, G)))^{M[G]}$

THEN $(\exists p \in G)(p \Vdash^* \varphi(\tau_1, \ldots, \tau_n))^M$.

BETTER VERSION OF LAST WEEK'S NOTES:

JUST FOR $\tau_1 = \tau_2$: $p \Vdash^* \tau_1 = \tau_2$ MEANS

(i) FOR ALL $\langle \pi_1, s_1 \rangle \in \tau_1$

$\overbrace{\{q \leq p : q \leq s_1 \to (\exists \langle \pi_2, s_2 \rangle \in \tau_2)(q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}}^{D(\pi_1, s_1, p)}$

(ii) FOR ALL $\langle \pi_2, s_2 \rangle \in \tau_2$

$\{q \leq p : q \leq s_2 \to (\exists \langle \pi_1, s_1 \rangle \in \tau_1)(q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2)\}$

ARE DENSE BELOW p.

THEN (i) GIVES $\tau_{1,G} \subseteq \tau_{2,G}$ AND (ii) GIVES $\tau_{2,G} \subseteq \tau_{1,G}$

① SO TAKE $x \in \tau_{1,G}$: THERE IS $\langle \pi_1, s_1 \rangle \in \tau_1$ WITH $s_1 \in G$ AND $x = VAL(\pi_1, G) = \pi_{1,G}$

THERE IS $r \in G$ WITH $r \leq p$ AND $r \leq s_1$

THEN $r \Vdash^* \tau_1 = \tau_2$ AND SO THERE IS $q \in G \cap D(\pi_1, s_1, r)$

BUT $q \leq r \leq s_1$ SO $(\exists \langle \pi_2, s_2 \rangle \in \tau_2)(q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)$

WE GET $s_2 \in G$, SO $\pi_{2,G} \in \tau_{2,G}$

AND $q \Vdash^* \pi_1 = \pi_2$

BY <u>INDUCTION</u> $\pi_{1,G} = \pi_{2,G}$ SO $x \in \tau_{2,G}$.

② ASSUME $\tau_{1,G} = \tau_{2,G}$

LET D BE THE UNION OF $\overbrace{\{r : r \Vdash^* \tau_1 = \tau_2\}}^{D_0}$,

$D_i := \{r : (\exists \langle \pi_1, s_1 \rangle \in \tau_1)(r \leq s_1 \wedge (\forall \langle \pi_2, s_2 \rangle \in \tau_2)(\forall q \in \mathbb{P})((q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2) \to q \perp r))\}$, AND

$D_{ii} := \{r : (\exists \langle \pi_2, s_2 \rangle \in \tau_2)(r \leq s_2 \wedge (\forall \langle \pi_1, s_1 \rangle \in \tau_1)(\forall q \in \mathbb{P})((q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2) \to q \perp r))\}$.

WE WANT G TO INTERSECT $\{r : r \Vdash^* \tau_1 = \tau_2\}$

IF $r \in G \cap D_i$ THEN WE GET $\langle \pi_1, s_1 \rangle \in \tau_1$ WITH $r \leq s_1$

AND SO $\pi_{1G} \in \tau_{1G}$, AND ALSO

IF $x \in \tau_{2G}$ THEN $x = \pi_{2,G}$ FOR SOME $\langle \pi_2, s_2 \rangle \in \tau_2$ WITH $s_2 \in G$

<u>IF</u> $\pi_{1G} = \pi_{2G}$ THEN BY INDUCTION THERE IS $p \in G$ SUCH THAT $p \Vdash^* \pi_1 = \pi_2$

BUT TAKE $q \in G$ WITH $q \leq p, r, s_2$

THEN $q \leq s_2$ AND $q \parallel r$ <u>SO</u> $q \not\Vdash^* \pi_1 = \pi_2$

BUT ALSO $q \leq p$ SO $q \Vdash^* \pi_1 = \pi_2$

<u>CONTRADICTION</u>

SIMILARLY $G \cap D_{ii} = \emptyset$

SO WE MUST SHOW $D_a \cup D_i \cup D_{ii}$ IS DENSE

LET $p \in \mathbb{P}$ AND ASSUME $p \notin D_a$ SO $p \not\Vdash^* \tau_1 = \tau_2$

SAY (i) FAILS; WE FIND $r \leq p$ THAT IS IN D_i

TAKE $\langle \pi_1, s_1 \rangle \in \tau_1$ AND USE "NOT DENSE BELOW p"

TO FIND $r \leq p$ SUCH THAT NO $q \leq r$ WORK AS IN (i)

SO $(\forall q \leq r)(q \leq s_1 \wedge (\forall \langle \pi_2, s_2 \rangle \in \tau_2)(\neg(q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)))$

IN PARTICULAR $r \leq s_1$

NOW IF $\langle \pi_2, s_2 \rangle \in \tau_2$ AND $q \leq s_2$ AND $q \Vdash^* \pi_1 = \pi_2$

THEN $q \perp r$

FOR IF $s \leq q, r$ THEN BOTH

$(s \leq s_2 \wedge s \Vdash^* \pi_1 = \pi_2)$ BECAUSE $s \leq q$ AND

$\neg(s \leq s_2 \wedge s \Vdash^* \pi_1 = \pi_2)$ BECAUSE $s \leq r$

SO $r \in D_i$

WHAT WE USE IN PRACTICE

① FOR ALL $p \in \mathbb{P}$

$p \Vdash \varphi(\tau_1, \ldots, \tau_n)$ IFF $(p \Vdash^* \varphi(\tau_1, \ldots, \tau_n))^M$

② FOR ALL M-GENERIC FILTERS G ON $\mathbb{P}$

$\varphi(\tau_{1,G}, \ldots, \tau_{n,G})^{M[G]}$ IFF $(\exists p \in G)(p \Vdash \varphi(\tau_1, \ldots, \tau_n))$

① DEFINABILITY OF FORCING ; ② TRUTH LEMMA

NOW EXPLAIN WHY ZFC HOLDS IN $M[G]$
(LAST PAGE OF LAST WEEK'S NOTES)

BACK TO $\neg$CH

COHEN STARTED FROM A MODEL M THAT SATISFIED GCH.

GÖDEL'S CONSTRUCTIBLE UNIVERSE [1938-40]

WITH COHEN WE TOOK $\mathbb{P} = Fn(\omega_2^M \times \omega, 2)$

FROM AN M-GENERIC FILTER G WE OBTAINED AN INJECTIVE MAP $f: \omega_2^M \longrightarrow \mathcal{P}^{M[G]}(\omega)$.

- FROM G TO $\bigcup G: \omega_2^M \times \omega \to 2$
- FROM $\bigcup G$ TO $f(\alpha) = \{n : \bigcup G(\alpha, n) = 1\}$
- $D_{\alpha,n} = \{p : \langle \alpha, n \rangle \in \mathrm{dom}\, p\}$ IS DENSE
- $E_{\alpha,\beta} = \{p : (\exists n)(\langle \alpha, n \rangle, \langle \beta, n \rangle \in \mathrm{dom}\, p \wedge p(\alpha,n) \neq p(\beta,n))\}$ IS DENSE IF $\alpha \neq \beta$.

GENERAL LEMMA:

LET $X, Y \in M$; LET $g: X \to Y$ BE A MAP IN $M[G]$

THERE IS A MAP $F: X \to [Y]^{\leq \omega}$ IN M

SUCH THAT $(\forall x \in X)(g(x) \in F(x))$.

"EVERY MAP IN $M[G]$ IS CAPTURED BY A COUNTABLE PIPE FROM M".

PROOF

TAKE A NAME γ IN M SUCH THAT $g = VAL(\gamma, G)$, AND $p \in G$ SUCH THAT

$$p \Vdash (\gamma : \check{X} \to \check{Y})$$

<u>DEFINE</u>, IN M, THE RELATION R_γ BY

$\langle q, \langle x, y \rangle \rangle \in R_\gamma$ IFF $q \in \mathbb{P}$ AND

$$q \leq p \to q \Vdash \gamma(\check{x}) = \check{y}$$
$$q \perp p \to y = y_0$$

WHERE y_0 IS A FIXED ELEMENT OF Y.

IF H IS M-GENERIC THEN $R_\gamma[H]$ IS A MAP FROM X TO Y:

- IF $p \notin H$ THEN $R_\gamma[H] = X \times \{y_0\}$
- IF $p \in H$ THEN $R_\gamma[H] =$ ----- A MAP.
- $R_\gamma[H] = VAL(\gamma, H)$

$\langle x, y \rangle \in R_\gamma[H] \iff (\exists q \in H)(\langle q, \langle x, y \rangle \rangle \in R_\gamma)$
$\iff (\exists q \in H)(q \Vdash \gamma(\check{x}) = \check{y})$
$\iff \gamma_G(x) = y$.

<u>DEFINE</u>, IN M, $F: X \to \mathcal{P}(Y)$ BY

$$F(x) = \{y : (\exists q \in \mathbb{P})(\langle q, \langle x, y \rangle \rangle \in R_\gamma)\}$$

CLEARLY $g(x) \in F(x)$ FOR ALL x (REALLY!)

- FOR ALL x THE SET $F(x)$ IS COUNTABLE

APPLY AC: $y \mapsto q_y$ FROM $F(x)$ TO $\mathbb{P}$

SUCH THAT $\langle q_y, \langle x, y \rangle \rangle \in R_\gamma$

OR $q_y \Vdash \gamma(\check{x}) = \check{y}$

IF $y \neq z$ THEN $q_y \perp q_z$

(IF $r \leq q_y, q_z$ THEN $r \Vdash \check{y} = \gamma(\check{x}) = \check{z}$)

SO $\{q_y : y \in F(x)\}$ IS AN ANTICHAIN

AND WE HAVE SEEN THAT

$$\left(\text{ANTICHAINS IN } Fn(\omega_2^M \times \omega, 2) \text{ ARE COUNTABLE}\right)^M$$

COROLLARY

FOR ALL $\alpha \in On \cap M$ WE HAVE $(\alpha \text{ IS A CARDINAL})^M$ IFF $(\alpha \text{ IS A CARDINAL})^{M[G]}$

PROOF

$\Leftarrow$ CLEAR IF $\beta < \alpha$ AND $f \in M$ IS A SURJECTION FROM β ONTO α THEN f IS ALSO A MEMBER OF $M[G]$ THAT IS A SURJECTION.

IN GENERAL "α IS A CARDINAL" IS DOWNWARD ABSOLUTE.

$\Rightarrow$ LET $\beta < \alpha$ AND LET $f \in M[G]$ BE A MAP FROM β TO α.

LET $F : \beta \to [\alpha]^{\leq \omega}$ BE AS IN THE LEMMA

THEN $|\bigcup\{F(\gamma) : \gamma < \beta\}|^M \leq |\beta|^M \cdot \aleph_0 < \alpha$.

WHY: WLOG $\alpha > \aleph_0$. BECAUSE THE NATURAL NUMBERS AND ω ARE ABSOLUTE.

EXERCISE

PROVE: $cf^M \alpha = cf^{M[G]} \alpha$ FOR ALL α.

LET'S CALCULATE $2^{\aleph_0}$ IN $M[G]$.

WE KNOW $2^{\aleph_0} \geq \omega_2^M$

ASSUME $val(\tau, G) \subseteq \omega$

LET $p \in G$ BE SUCH THAT $p \Vdash \tau \subseteq \check{\omega}$

MAKE A RELATION AGAIN (OR ANOTHER NAME)

$$\tau' = \{ \langle \check{n}, q \rangle : (q \leq p \to q \Vdash \check{n} \in \tau) \wedge (q \perp p \to n = 0) \}$$

THEN $\emptyset \Vdash \tau' \subseteq \check{\omega}$

$p \Vdash \tau' = \tau$ AND $val(\tau, G) = val(\tau', G)$

SO ALL SUBSETS OF ω COME FROM RELATIVELY NICE NAMES: SUBSETS OF $\{\check{n} : n \in \omega\} \times \mathbb{P}$.

PROPERTIES OF SUCH A SUBSET/NAME τ':

IF $\langle \check{n}, q \rangle \in \tau$ AND $r \leq q$ THEN $\langle \check{n}, r \rangle \in \tau'$

BECAUSE IF $q \Vdash \check{n} \in \tau$ THEN $r \Vdash \check{n} \in \tau$

MAKE τ' EVEN NICER: TAKE, FOR EVERY n, A MAXIMAL ANTICHAIN A_n IN $\{q : \langle \check{n}, q \rangle \in \tau'\}$

AND LET $\tau'' = \bigcup_{n \in \omega} \{\check{n}\} \times A_n$.

NOW WE HAVE

- τ'' IS A COUNTABLE SET
- $\mathrm{VAL}(\tau'', G) = \mathrm{VAL}(\tau', G) = \mathrm{VAL}(\tau, G)$.

SO HOW MANY SUCH NAMES ARE THERE?

WELL, AT MOST

$$\left| \left[\omega \times \mathrm{Fn}(\omega_2^M \times \omega, 2)\right]^{\leq \aleph_0} \right|^M$$

BUT $\left(|\mathrm{Fn}(\omega_2^M \times \omega, 2)| = \aleph_2\right)^M$

SO $\left| \left[\omega \times \mathrm{Fn}(\omega_2^M \times \omega, 2)\right]^{\leq \aleph_0} \right|^M \leq \left(\aleph_2^{\aleph_0}\right)^M = \aleph_2^M$

BY THE GCH IN M

WE CONCLUDE $\left(2^{\aleph_0} = \aleph_2\right)^{M[G]}$.

EXERCISE $\left(2^{\aleph_1} = \aleph_2\right)^{M[G]}$

SO $2^{\aleph_0} = 2^{\aleph_1}$ IS CONSISTENT

SOLOVAY [1965]:

$2^{\aleph_0}$ CAN BE ANYTHING IT OUGHT TO BE

LET κ BE AN INFINITE CARDINAL OF M SUCH THAT $\mathrm{cf}\,\kappa > \aleph_0$ IN M.

THEN THERE IS AN EXCELLENT EXTENSION N OF M IN WHICH $2^{\aleph_0} = \kappa$.

[THIS MODEL M SATISFIES GCH]

EXCELLENT: - $\mathrm{On} \cap M = \mathrm{On} \cap N$

- $\mathrm{cf}^M \alpha = \mathrm{cf}^N \alpha$ FOR ALL α.

HOW: TAKE G TO BE M-GENERIC ON $\mathrm{Fn}(\kappa \times \omega, 2)$

EVERYTHING GOES THROUGH AS IN THE CASE $\kappa = \omega_2^M$.

YOU GET $2^{\lambda} = \kappa$ FOR ALL $\lambda < \kappa$.

EXERCISE

- IF $\kappa = \aleph_n$ FOR SOME $n \in \omega \setminus \{1, 0\}$
  THEN - $2^{\aleph_m} = \aleph_n$ FOR $m \in n$
  - $2^{\aleph_m} = \aleph_{m+1}$ FOR $m \geq n$.
- IF $\kappa = \aleph_{\omega+1}$ THEN $2^{\aleph_n} = \aleph_{\omega+1}$ FOR ALL $n \in \omega$
  AND SO $2^{\aleph_\omega} = \aleph_{\omega+1}$ AS WELL
- WHAT HAPPENS TO $2^{\aleph_0}$ IF WE USE $\mathrm{Fn}(\aleph_\omega \times \omega, 2)$?

SO WE HAVE THE FOLLOWING CONSISTENCIES

- GÖDEL : AC + GCH
- COHEN: $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$
  [CALCULATE $2^{\aleph_\alpha}$ FOR $\alpha \geq 2$]
- COHEN/SOLOVAY $2^{\aleph_0} = \aleph_\alpha$ AS LONG AS $cf\,\alpha > \aleph_0$

WHAT ELSE?

- LET κ BE REGULAR AND λ SUCH THAT $cf\,\lambda > \kappa$. THEN WE CAN ACHIEVE
  - $2^{\kappa} = \lambda$
  - $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ IF $\aleph_\alpha < \kappa$. [SOLOVAY, EXCELLENT]

FOR DEFINITENESS LET $\kappa = \aleph_1$.

HOMEWORK 12 (42)

LET $\mathbb{P} = Fn(\lambda \times \omega_1, 2, \aleph_1)$

THE SET OF FUNCTIONS p WITH $\mathrm{dom}\, p \in [\lambda \times \omega_1]^{\leq \aleph_0}$ AND $\mathrm{ran}\, p \subseteq 2$.

ORDERED AS BEFORE: $p \leq q$ IF $p \supseteq q$.

HOMEWORK: ALL ANTICHAINS HAVE CARDINALITY AT MOST $\aleph_1$ [SOLOVAY STARTED FROM GCH]

SO EVERY MAP $g : X \to Y$ WITH $X, Y \in M$ IS CAPTURED BY A PIPE $F : X \to [Y]^{\leq \aleph_1}$

SO, IF $cf\,\alpha > \aleph_1$ IN M THEN $cf\,\alpha > \aleph_1$ IN $M[G]$.

WHAT IF $cf\,\alpha \leq \aleph_1$?? CAN WE GET $cf\,\alpha = \aleph_0$?

NEW PHENOMENON:

IF $g : \omega \to X$ WITH $X \in M$ AND $g \in M[G]$ THEN $g \in M$.

LET γ BE A NAME FOR g, SO $g = \mathrm{val}(\gamma, G)$

LET $p \in G$ BE SUCH THAT $p \Vdash \gamma : \check{\omega} \to \check{X}$

LET $q \leq p$.

THEN $q \Vdash (\exists z)(z \in \check{X} \wedge \gamma(\check{\delta}) = z)$

SO THERE ARE $r \leq q$ AND σ SUCH THAT $r \Vdash (\sigma \in \check{X} \wedge \gamma(\check{\delta}) = \sigma)$

BUT THEN THERE ARE $s \leq r$ AND $\langle \check{x}, \mathbb{1} \rangle \in \check{X}$ SUCH THAT $s \Vdash (\check{x} = \sigma \wedge \gamma(\check{\delta}) = \sigma)$

OR $\underline{s \Vdash (\gamma(\check{\delta}) = \check{x})}$

RECURSIVELY:

TAKE $q_0 \leq q$ AND $x_0 \in X$

SUCH THAT $q_0 \Vdash \gamma(\check{0}) = \check{x}_0$

AND $q_{n+1} \leq q_n$ AND $x_{n+1} \in X$

SUCH THAT $q_{n+1} \Vdash \gamma(\check{n+1}) = \check{x}_{n+1}$

EACH q_n IS A COUNTABLE FUNCTION

AND SO $r = \bigcup_{n \in \omega} q_n$ IS A COUNTABLE FUNCTION TOO

WE HAVE $r \in Fn(\lambda \times \omega_1, 2, \aleph_1)$

AND FOR ALL n $r \Vdash \gamma(\check{n}) = \check{x}_n$

DEFINE $f = \{\langle n, x_n \rangle : n \in \omega\}$ NOTE $f \in M$

THEN $r \Vdash (\gamma = \check{f})$

CONCLUSION $D_\gamma = \{r \leq p : (\exists f \in X^\omega)(r \Vdash \gamma = \check{f})\}$

IS DENSE BELOW p.

TAKE $r \in D_\gamma \cap G$ WITH ITS f

THEN $g = f \in M$

SO: $\mathcal{P}^M(\omega) = \mathcal{P}^{M[G]}(\omega)$ WE STILL HAVE $2^{\aleph_0} = \aleph_1$

. IF $cf^M \alpha = \aleph_1$ THEN $cf^{M[G]} \alpha = \aleph_1$

NO NEW FUNCTIONS FROM ω TO α.

TO SEE $2^{\aleph_1} = \lambda$ DO THE SAME COMPUTATIONS AS FOR THE CASE OF $Fn(\omega_2 \times \omega, 2)$

- $2^{\aleph_1} \geq \lambda$ BECAUSE G GIVES AN INJECTION FROM λ INTO $\mathcal{P}^{M[G]}(\omega_1)$
- $2^{\aleph_1} \leq \lambda$ USE ANTICHAINS AGAIN [AND GCH]
  - $|Fn(\lambda \times \omega_1, 2, \aleph_1)| \leq \lambda^{\aleph_0} = \lambda$
  - $\left|[Fn(\lambda \times \omega_1, 2, \aleph_1)]^{\leq \aleph_1}\right| \leq \lambda^{\aleph_1} = \lambda$.