SET THEORY 2021-12-06
[TOMORROW IS ANOTHER HISTORIC DATE]

OUR GOALS FOR TODAY:
- CONSTRUCT $M[G]$ FROM $M \cup \{G\}$.
- SHOW HOW TO MAKE RELATIONS LIKE R
- PROVE $M[G]$ SATISFIES (ENOUGH OF) ZFC

WE DO THIS IN A GENERAL CONTEXT
WE TAKE A PARTIALLY ORDERED SET $\langle \mathbb{P}, \leq \rangle$ IN M
YOU CAN THINK OF $Fn(\omega_2^M \times \omega, 2)$ ALL THE TIME,
BUT NOTHING WILL BE SPECIFIC FOR THAT SITUATION.
SO: $\leq$ IS A RELATION ON $\mathbb{P}$ THAT SATISFIES
- $x \leq x$; - $x \leq y \wedge y \leq x \to x = y$;
- $x \leq y \wedge y \leq z \to x \leq z$.

WE ADD, FOR TECHNICAL REASONS: $\mathbb{P}$ HAS A MAXIMUM, WRITTEN $\mathbb{1}_{\mathbb{P}}$ (OR JUST $\mathbb{1}$)
[IN $Fn(\omega_2^M \times \omega, 2)$ THE MAXIMUM IS $\emptyset$ THE EMPTY FUNCTION.]

WE CALL p AND q IN $\mathbb{P}$ <u>COMPATIBLE</u> IF THERE IS AN r IN $\mathbb{P}$ SUCH THAT $r \leq p, q$
AND <u>INCOMPATIBLE</u> IF THERE IS NO SUCH r

NOTATION $p \perp q$ FOR INCOMPATIBLE
$p \parallel q$ FOR COMPATIBLE

[IN $Fn(\omega_2^M \times \omega, 2)$ COMPATIBLE: $p \cup q$ IS A FUNCTION,
I.E. IF $x \in dom\, p \cap dom\, q$ THEN $p(x) = q(x)$
INCOMPATIBLE $p \cup q$ IS NOT A FUNCTION
I.E. $p(x) \neq q(x)$ FOR SOME $x \in dom\, p \cap dom\, q$]

WE ALREADY KNOW [M IS COUNTABLE]:
FOR EVERY $p \in \mathbb{P}$ THERE IS A FILTER G SUCH THAT $p \in G$ AND $G \cap D \neq \emptyset$ FOR ALL DENSE SUBSETS IN $\mathbb{P}$ THAT ARE IN M.

FILTER $p, q \in G \to (\exists r \in G)(r \leq p, q)$; $p \in G \wedge p \leq q \to q \in G$
DENSE $\forall p \in \mathbb{P}\ \exists q \in D\ q \leq p$
WE CALL G AN M-GENERIC FILTER
IF $(\forall D \in M)(D \text{ DENSE IN } \mathbb{P} \to D \cap G \neq \emptyset)$

PROOF $\langle D_m : m \in \omega \rangle$ ENUMERATES $\{D : D \text{ DENSE IN } \mathbb{P}\}^M$
- $p_0 \in D_0$ $p_0 \leq p$; $p_{m+1} \in D_{m+1}$ $p_{m+1} \leq p_m$
- $G = \{q : (\exists m)(p_m \leq q)\}$.

WE MAKE THE EXISTENCE OF RELATIONS LIKE R ALMOST TRUE BY DEFINITION WHEN WE SHOW HOW TO BUILD $M[G]$ OUT OF $M \cup \{G\}$

SHOENFIELD

DEFINE $\in_G$ ON M: $a \in_G b$ MEANS

$$(\exists p \in G)(\langle a, p\rangle \in b)$$

SO $a \in_G b$ IMPLIES $\rho(a) < \rho(b)$ (RANK)

DEFINE $\mathrm{SET}_G(b) = \{\mathrm{SET}_G(a) : a \in_G b\}$

RECURSION ON RANK.

$M[G] = \{\mathrm{SET}_G(b) : b \in M\}$

$G = \mathrm{SET}_G(\Gamma)$ WHERE $\Gamma = \{\langle \check{p}, p\rangle : p \in \mathbb{P}\}$

AND $\check{b} = \{\langle \check{a}, \mathbb{1}\rangle : a \in b\}$

SO FIRST: $\mathrm{SET}_G(\check{b}) = b$ $(b \in M)$

SECOND $\mathrm{SET}_G(\Gamma) = G$

SOME (MANY?) PEOPLE DON'T LIKE THIS BECAUSE $a \in_G b$ HARDLY EVER HOLDS FOR 'NORMAL' SETS

UNIVERSAL TERMINOLOGY: "b IS A NAME FOR $\mathrm{SET}_G(b)$"

WE DEFINE 'PROPER' $\mathbb{P}$-NAMES; BY RECURSION

WE FOLLOW KUNEN'S BOOKS 1980 p188, 2011 p246

τ IS A $\mathbb{P}$-NAME IFF
- τ IS A RELATION
- $(\forall \langle \sigma, p\rangle \in \tau)(p \in \mathbb{P} \wedge \sigma$ IS A $\mathbb{P}$-NAME$)$

RECURSION ON THE WELL-FOUNDED RELATION

$x \mathrel{E} y$: $x \in \mathrm{TRCL}(y)$

STILL $\check{x} = \{\langle \check{y}, \mathbb{1}\rangle : y \in x\}$ IS A $\mathbb{P}$-NAME

$\Gamma = \{\langle \check{p}, p\rangle : p \in \mathbb{P}\}$ IS A $\mathbb{P}$-NAME

- $\tau \in M$ IS A $\mathbb{P}$-NAME IN M IF IT IS A $\mathbb{P}$-NAME [USE TRANSITIVITY]
- $M^{\mathbb{P}} = \{\tau \in M : \tau$ IS A $\mathbb{P}$-NAME$\}$

$(V^{\mathbb{P}} = \{\tau : \tau$ IS A $\mathbb{P}$-NAME$\})$

FOR NAMES:

$$VAL(\tau, G) = \{ VAL(\sigma, G) : (\exists p \in G)(\langle \sigma, p \rangle \in \tau) \}$$

[WE USE $\in_G$ ONLY FOR NAMES]

$$M[G] = \{ VAL(\tau, G) : \tau \in M^{\mathbb{P}} \}$$

$VAL(\check{x}, G) = x \qquad x \in M$

$VAL(\dot{\Pi}, G) = G$

SO $M \cup \{G\} \subseteq M[G]$

- $\emptyset$ IS A NAME: $VAL(\emptyset, G) = \emptyset$
- $\{\langle \emptyset, p \rangle\}$ IS A $\mathbb{P}$-NAME IF $p \in \mathbb{P}$

$$VAL(\{\langle \emptyset, p \rangle\}, G) = \begin{cases} \{\emptyset\} & p \in G \\ \emptyset & p \notin G \end{cases}$$

A PARTIALLY ORDERED SET $\langle \mathbb{P}, \leq \rangle$ IS <u>SEPARATIVE</u> IF $p \not\leq q$ IMPLIES THERE IS $r \leq p$ WITH $r \perp q$

$Fn(\omega_2^M \times \omega, 2)$ IS SEPARATIVE

$p \not\leq q$ MEANS $q \not\subseteq p$

SO <u>EITHER</u> THERE IS $x \in dom(q) \setminus dom p$

<u>IN THAT CASE</u> $r = p \cup \{\langle x, 1 - q(x) \rangle\}$ WORKS

<u>OR</u> $dom\, q \subseteq dom\, p$

BUT THEN $q(x) \neq p(x)$ FOR SOME x AND ALREADY $p \perp q$.

SO, IF $q < p$ THEN THERE IS $r \leq p$ SUCH THAT $r \perp q$.

IN SEPARATIVE PARTIAL ORDERS $VAL(\tau, G)$ DEPENDS ON G.

- $M[G]$ IS TRANSITIVE (BY DEFINITION)
- $\rho(VAL(\tau, G)) \leq \rho(\tau)$
- $On \cap M[G] = On \cap M$
- $up(\sigma, \tau) = \{ \langle \sigma, \mathbb{1} \rangle, \langle \tau, \mathbb{1} \rangle \}$

  $op(\sigma, \tau) = up(up(\sigma, \sigma), up(\sigma, \tau))$

  THESE ARE NAMES AND

  $VAL(up(\sigma, \tau), G) = \{ VAL(\sigma, G), VAL(\tau, G) \}$

  $VAL(op(\sigma, \tau), G) = \langle VAL(\sigma, G), VAL(\tau, G) \rangle$

ABBREVIATION <u>$\tau_G = VAL(\tau, G)$</u>

FIRST RESULT: $M[G]$ SATISFIES EXTENSIONALITY, REGULARITY, PAIRING, UNION

UNION: GIVEN τ LET $\pi = \bigcup \operatorname{dom} \tau$

CHECK $\bigcup \tau_G \subseteq \pi_G$

→ TRY TO DEFINE σ SUCH THAT ALWAYS
$$\bigcup \tau_G = \sigma_G$$

EXERCISE ON DENSITY

- $E \subseteq \mathbb{P}$ IS DENSE BELOW p IF $(\forall q \leq p)(\exists r \leq q)(r \in E)$
- IF $E \subseteq \mathbb{P}$ AND $E \in M$ THEN
$$D_E = \{ p : (\exists r \in E)(r \leq p) \vee (\forall r \in E)(r \perp p) \}$$
IS DENSE.
- IF G IS GENERIC THEN $G \cap D_E \neq \emptyset$
SO ... $G \cap E \neq \emptyset \;\vee\; (\exists q \in G)(\forall r \in E)(r \perp q)$
- IF G IS GENERIC, $p \in G$, AND E IS DENSE BELOW p THEN $G \cap E \neq \emptyset$.

ALSO IF $\mathbb{P}$ IS SEPARATIVE AND G IS M-GENERIC THEN $G \notin M$.

FORCING

WAIT! IN $M[G]$ WITH G GENERIC ON $Fn(\omega_2^M \times \omega, 2)$ DO WE HAVE $\bigcup G \in M$?

YES $\Psi = \{ \langle \langle \langle \alpha, n \rangle, i \rangle^\vee, p \rangle : p \in \mathbb{P};\ \langle \alpha, n \rangle \in \operatorname{dom} p;\ p(\alpha, n) = i \}$

CALCULATE $\Psi_G = \bigcup G$

ALSO $f : \omega_2^M \to \mathcal{P}^{M[G]}(\omega)$ HAS A NAME

$f(\alpha) = \{ n : \bigcup G(n) = 1 \}$

$f = \{ \langle \alpha, f(\alpha) \rangle : \alpha \in \omega_2^M \}$
$\quad = \{ \operatorname{VAL}(\operatorname{op}(\check{\alpha}, \tau_\alpha), G) : \alpha \in \omega_2^M \}$

$\tau_\alpha = \{ \langle \check{n}, p \rangle : \langle \alpha, n \rangle \in \operatorname{dom} p \wedge p(\alpha, n) = 1 \}$

SO $\Phi = \{ \langle \operatorname{op}(\check{\alpha}, \tau_\alpha), \mathbb{1} \rangle : \alpha \in \omega_2^M \}$ IS A NAME FOR f.

HOMEWORK: $f(\alpha) \cap x$ AND $x \setminus f(\alpha)$ INFINITE IF $x \in M$ IS INFINITE

CAN WE SEE THIS FROM M?

IF $\{\langle\langle 0,10\rangle,1\rangle\} \in G$ THEN $0 \in f(0)$

IF $\{\langle\langle 0,10\rangle,0\rangle\} \in G$ THEN $0 \notin f(0)$

IF $q=\{\langle\langle\alpha,10\rangle,1\rangle,\langle\langle\alpha+1,10\rangle,0\rangle\} \in G$ THEN

$10 \in f(\alpha)\setminus f(\alpha+1)$

THUS WE GET (PARTIAL) INFO ABOUT $M[G]$ in M

WE SHALL SAY $\swarrow$ FORCES

$$\{\langle\langle 0,10\rangle,1\rangle\} \Vdash \check{0} \in \Phi(\check{0})$$

$$\{\langle\langle 0,10\rangle,0\rangle\} \Vdash \check{0} \notin \Phi(\check{0})$$

$$q \Vdash \check{10} \in \Phi(\check{\alpha})\setminus\Phi(\check{\alpha+1})$$

DEFINITION

IN GENERAL IF φ IS A FORMULA AND $\tau_1,\ldots,\tau_k$ ARE NAMES THEN

$$p \Vdash \varphi(\tau_1,\ldots,\tau_k) \text{ IFF}$$

FOR ALL M-GENERIC G WITH $p\in G$ WE HAVE

$$M[G] \models \varphi(\mathrm{VAL}(\tau_1,G),\ldots,\mathrm{VAL}(\tau_k,G))$$

REMEMBER OUR FUNCTIONS $g:\omega\to\omega_1^M$ AND $h:\omega_1^M\to\omega_2^M$?

EACH HAS A NAME. SAY σ AND τ

WHAT WOULD BE OUR RELATION R?

FOR g WE TAKE

$$R_g = \{\langle\langle n,\alpha\rangle,p\rangle : n\in\omega,\ \alpha\in\omega_1,\ p\in\mathbb{P} \text{ AND } p\Vdash\sigma(\check{n})=\check{\alpha}\}$$

① IS R_g IN M?

② DOES IT ACTUALLY WORK?

HOW DO WE KNOW THAT $R_g[G]$ IS A FUNCTION?

① THAT IS OUR NEXT JOB

② YES, WELL, ..., ALMOST

WE DEFINE $\Vdash^*$ WITHOUT MENTIONING G
AND PROVE $p \Vdash \varphi$ IFF $(p \Vdash^* \varphi)^M$
NOTE
$p \Vdash^* \varphi$ CAN BE DECIDED COMPLETELY WITHIN M
SO IT MAKES R_φ WELL-DEFINED <u>IN M</u>

START WITH EQUALITY

$p \Vdash^* \tau_1 = \tau_2$ IFF

- FOR ALL $\langle \pi_1, s_1 \rangle \in \tau_1$
  $\{q \le p : q \le s_1 \to (\exists \langle \pi_2, s_2\rangle \in \tau_2)(q \le s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$
  $(q \Vdash^* \pi_1 \in \tau_1)$ " π_1 IS EQUAL TO SOME π_2 IN τ_2"
  IS DENSE BELOW p.
- AND CONVERSELY
  FOR ALL $\langle \pi_2, s_2 \rangle \in \tau_2$
  $\{q \le p : q \le s_2 \to (\exists \langle \pi_1, s_1\rangle \in \tau_1)(q \le s_1 \wedge q \Vdash^* \pi_1 = \pi_2)\}$
  IS DENSE BELOW p

RECURSION ON PAIRS OF NAMES
$\langle \pi_1, \pi_2 \rangle \ E \ \langle \tau_1, \tau_2 \rangle$ IFF $\pi_1 \in \operatorname{dom} \tau_1 \wedge \pi_2 \in \operatorname{dom} \tau_2$
THIS IS WELL-FOUNDED
AND WE GET $F(\tau_1, \tau_2) = \{p \in \mathbb{P} : p \Vdash^* \tau_1 = \tau_2\}$
THIS IS ABSOLUTE

NOW MEMBERSHIP

$p \Vdash^* \tau_1 \in \tau_2$ IFF
$\{q \le p : (\exists \langle \pi, s \rangle \in \tau_2)(q \le s \wedge q \Vdash^* \pi = \tau_1)\}$
IS DENSE BELOW p.
STILL ABSOLUTE

- CONJUNCTION $p \Vdash^* (\varphi(\tau_1, \dots, \tau_n) \wedge \psi(\tau_1, \dots, \tau_n))$
  IFF $p \Vdash^* \varphi(\tau_1, \dots, \tau_n)$ <u>AND</u> $p \Vdash^* \psi(\tau_1, \dots, \tau_n)$
- NEGATION $p \Vdash^* \neg\varphi(\tau_1, \dots, \tau_n)$ IFF THERE IS <u>NO</u> $q \le p$ SUCH THAT $q \Vdash^* \varphi(\tau_1, \dots, \tau_n)$

QUANTIFICATION

$p \Vdash^* (\exists x)\varphi(x,\tau_1,\dots,\tau_n)$ IFF

$$\{q \le p : (\exists \sigma \in V^{\mathbb{P}})(q \Vdash^* \varphi(\sigma,\tau_1,\dots,\tau_n))\}$$

IS DENSE BELOW p.

[NO LONGER ABSOLUTE]

EXERCISE EQUIVALENT ARE

① $p \Vdash^* \varphi$

② $(\forall r \le p)(r \Vdash^* \varphi)$

③ $\{r : r \Vdash^* \varphi\}$ IS DENSE BELOW p

② → ① ② → ③ CLEAR

① → ② ③ → ② INDUCTION ON COMPLEXITY

BIG THEOREM

M A TRANSITIVE MODEL OF ZFC

$\mathbb{P}$ A PARTIAL ORDER IN M

$\tau_1,\dots,\tau_n \in M^{\mathbb{P}}$

G AN M-GENERIC FILTER ON $\mathbb{P}$

$\varphi(x_1,\dots,x_n)$ FORMULA WITH FREE VARIABLES SHOWN

THEN

- IF $p \in G$ AND $(p \Vdash^* \varphi(\tau_1,\dots,\tau_n))^M$
  THEN $(\varphi(val(\tau_1,G),\dots,val(\tau_n,G)))^{M[G]}$
- IF $(\varphi(val(\tau_1,G),\dots,val(\tau_n,G)))^{M[G]}$
  THEN THERE IS A $p \in G$ SUCH THAT
  $(p \Vdash^* \varphi(\tau_1,\dots,\tau_n))^M$

INDUCTION ON THE COMPLEXITY OF φ

- $\tau_1 = \tau_2$ [INDUCTION ON E]

ASSUME $p \Vdash^* \tau_1 = \tau_2$ AND $p \in G$ ②

WE GET ① $val(\tau_1,G) \subseteq val(\tau_2,G)$ AND $val(\tau_2,G) \subseteq val(\tau_1,G)$

① TAKE $\langle \pi_1, s_1\rangle \in \tau_1$ WITH $s_1 \in G$ & $val(\pi_1,G) \in val(\tau_1,G)$

TAKE $r \in G$ WITH $r \le p, s_1$: $r \Vdash^* \tau_1 = \tau_2$

THERE IS $q \in G$ SUCH THAT $q \le r$ AND

$q \le s_1 \rightarrow (\exists \langle \pi_2, s_2\rangle \in \tau_2)(q \le s_2 \wedge q \Vdash^* \pi_1 = \pi_2)$

BUT $q \le s_1$ HENCE WE CAN TAKE $\langle \pi_2, s_2 \rangle$

THEN $s_2 \in G$ SO $val(\pi_2,G) \in val(\tau_2,G)$

INDUCTIVE ASSUMPTION BECAUSE $q \Vdash^* \pi_1 = \pi_2$

WE GET $val(\pi_1,G) = val(\pi_2,G)$

DONE!

CONVERSE: LET D BE THE UNION OF

- $D_1 = \{ r \in \mathbb{P} : r \Vdash^* \tau_1 = \tau_2 \}$
- $D_2 = \{ r \in \mathbb{P} : (\exists \langle \pi_1, s_1 \rangle \in \tau_1)(r \leq s_1 \wedge (\forall \langle \pi_2, s_2 \rangle \in \tau_2)(\forall q \in \mathbb{P})((q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2) \rightarrow q \perp r))) \}$
- $D_3 = \{ r \in \mathbb{P} : (\exists \langle \pi_2, s_2 \rangle \in \tau_2)(r \leq s_2 \wedge (\forall \langle \pi_1, s_1 \rangle \in \tau_1)(\forall q \in \mathbb{P})((q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2) \rightarrow q \perp r))) \}$

CLAIM: D IS DENSE AND $D \in M$

- $D \in M$: CLEAR
- D DENSE: LET $p \in \mathbb{P}$

$p \Vdash^* \tau_1 = \tau_2$ OR $p \not\Vdash^* \tau_1 = \tau_2$

ASSUME THE LATTER: ONE OF OUR TWO CONDITIONS FAILS

SAY THERE IS A $\langle \pi_2, s_2 \rangle \in \tau_2$ SUCH THAT $\{ q \leq p : q \leq s_2 \rightarrow (\exists \langle \pi_1, s_1 \rangle \in \tau_1)(q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2) \}$

IS NOT DENSE BELOW p

SO WE HAVE $r \leq p$ SUCH THAT THE IMPLICATION FAILS FOR ALL $q \leq r$:

$$q \leq s_2 \wedge (\forall \langle \pi_1, s_1 \rangle \in \tau_1)(\neg(q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2))$$

SO $r \leq s_2 \wedge \forall \langle \pi_1, s_1 \rangle \in \tau_1 \, \forall q \in \mathbb{P} [q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2 \rightarrow q \perp r]$

So $G \cap D \neq \emptyset$

$p \in G \cap D_3$: $\langle \pi_2, s_2 \rangle \in \tau_2$, $r \leq s_2$

SO $s_2 \in G$ AND $\pi_{2,G} \in \tau_{2,G}$

IF $\pi_{2,G} = \pi_{1,G}$ FOR SOME $\langle \pi_1, s_1 \rangle \in \tau_1$ WITH $s_1 \in G$

THEN THERE IS $q_0 \in G$ SUCH THAT $q_0 \Vdash^* \pi_1 = \pi_2$

TAKE $q \leq r, s_2, s_1$

THEN $q \Vdash^* \pi_1 = \pi_2$ SO $q \perp r$ CONTRADICTION

So $G \cap D_3 = \emptyset$ LIKEWISE $G \cap D_2 = \emptyset$ AND

SO $G \cap D_1 \neq \emptyset$: THERE IS $p \in G$ SUCH THAT $p \Vdash^* \tau_1 = \tau_2$.

THE WHOLE PROOF CAN BE FOUND IN KUNEN'S BOOK

MAIN RESULT OF TODAY:

GIVEN A COUNTABLE MODEL M, $\mathbb{P} \in M$ A PARTIAL ORDER
$\varphi(x_1, \dots, x_k)$ A FORMULA WITH FREE VARIABLES SHOWN
LET $\tau_1, \dots, \tau_k \in M^{\mathbb{P}}$

① FOR $p \in \mathbb{P}$
$$p \Vdash \varphi(\tau_1, \dots, \tau_k) \text{ IFF } (p \Vdash^* \varphi(\tau_1, \dots, \tau_k))^M$$

② FOR ALL M-GENERIC FILTERS ON $\mathbb{P}$
$$\varphi(\tau_{1G}, \dots, \tau_{k,G})^{M[G]} \text{ IFF } (\exists p \in G)(p \Vdash \varphi(\tau_1, \dots, \tau_k))$$

①. PREVIOUS THEOREM PLUS DEFINITION OF $\Vdash$ GIVES $\Vdash^*$ IMPLIES $\Vdash$

CONVERSE: $\{r : (r \Vdash^* \varphi(\tau_1, \dots, \tau_k))^M\}$ DENSE BELOW p

IF NOT TAKE $q \leq p$ WITH $\{r \leq q : (r \Vdash^* \varphi(\tau_1, \dots, \tau_k))^M\}$ EMPTY.

THEN $(q \Vdash^* \neg\varphi(\tau_1, \dots, \tau_k))^M$

AND SO $q \Vdash \neg\varphi(\tau_1, \dots, \tau_k)$

BUT IF G IS GENERIC AND $q \in G$ THEN ALSO $p \in G$ AND $(\neg\varphi(\tau_{1G}, \dots, \tau_{kG}))^{M[G]}$ CONTRADICTION

② LEFT TO RIGHT: THEOREM PLUS ①

RIGHT TO LEFT: DEFINITION OF $\Vdash$

<u>NOW</u> WE CAN <u>DEFINE</u> R_g:

IF $g : \omega \to \omega_1^M$ IN $M[G]$

THEN $g = \gamma_G$ FOR SOME $\gamma \in M^{\mathbb{P}}$.

AND $(\gamma_G \text{ IS A FUNCTION, } \operatorname{dom} \gamma = \omega, \operatorname{ran} \gamma \subseteq \omega_1^M)^{M[G]}$

SO THERE IS A $p \in G$ SUCH THAT
$$p \Vdash \text{"}\gamma \text{ IS A FUNCTION, } \operatorname{dom} \gamma = \omega, \operatorname{ran} \gamma \subseteq \omega_1\text{"}$$
OR $$p \Vdash^* \text{"}\gamma \text{ IS A FUNCTION, } \operatorname{dom} \gamma = \omega, \operatorname{ran} \gamma \subseteq \omega_1\text{"}$$

<u>DEFINE</u> R_γ AS FOLLOWS

PART 1: $\{\langle q, \langle n, \alpha\rangle\rangle : q \leq p \wedge (q \Vdash^* \gamma(\check{n}) = \check{\alpha})^M\}$

PART 2: $\{\langle q, \langle n, 0\rangle\rangle : q \perp p \wedge n \in \omega\}$

FOR EVERY M-GENERIC H THE VALUE γ_H IS A FUNCTION: IF $p \in H$ BECAUSE $p \Vdash$ "γ IS A FUNCTION" IF $p \notin H$ IT IS THE ZERO FUNCTION.

IT REMAINS TO SHOW THAT $M[G]$ IS A MODEL OF ZFC.

- EXTENSIONALITY + FOUNDATION: TRANSITIVITY
- PAIRING + UNION [HOMEWORK ON GI]
- INFINITY: $\omega \in M[G]$: $\omega = \check{\omega}_G$
- POWER SET [HOMEWORK ON GI]
- AC: LET $\sigma \in M^{\mathbb{P}}$

  LET $\langle \pi_\gamma : \gamma < \alpha \rangle$ ENUMERATE $\operatorname{dom} \sigma$ [AC]

  LET $\tau = \{ op(\check{\gamma}, \pi_\gamma) : \gamma \in \alpha \} \times \{\mathbb{1}\}$

  SO $\tau \in M^{\mathbb{P}}$ AND $\tau_G = \{ \langle \gamma, \pi_{\gamma,G} \rangle : \gamma < \alpha \}$

  WE FIND A FUNCTION, τ_G, IN $M[G]$

  WITH - $\operatorname{dom} \tau_G = \alpha$

  - $\sigma_G \subseteq \operatorname{ran} \tau_G$

  THAT IS ENOUGH TO WELL-ORDER σ_G.

- SEPARATION/COMPREHENSION

  GIVEN $\sigma, \tau_1, \dots, \tau_n \in M^{\mathbb{P}}$ AND $\varphi(x, v, y_1, \dots, y_n)$

  LET $\varrho = \{ \langle \pi, p \rangle \in \operatorname{dom}(\sigma) \times \mathbb{P} : p \Vdash (\pi \in \sigma \wedge \varphi(\pi, \sigma, \underbrace{\tau_1, \dots, \tau_n}_{\tau})) \}$

  THEN $\varrho_G = \{ a \in \sigma_G : \varphi(a, \sigma_G, \tau_G)^{M[G]} \}$

  IF $a \in \varrho_G$ THEN THERE IS $\langle \pi, p \rangle$ SUCH THAT $\pi_G = a$, $p \in G$ AND $p \Vdash (\pi \in \sigma \wedge \varphi(\pi, \sigma, \tau))$

  SO $a \in \sigma_G \wedge \varphi(\underset{a}{\pi_G}, \sigma_G, \tau_G)^{M[G]}$

  CONVERSELY IF $a \in \sigma_G \wedge \varphi(a, \sigma_G, \tau_G)^{M[G]}$

  THEN $a = \pi_G$ FOR SOME $\pi \in \operatorname{dom}(\sigma)$

  AND THERE IS $p \in G$ SUCH THAT

  $p \Vdash \pi \in \sigma \wedge \varphi(\pi, \sigma, \tau)$

  SO $\langle \pi, p \rangle \in \varrho$ AND $\pi_G \in \varrho_G$ AS $p \in G$

- REPLACEMENT

  GIVEN $\varphi(x, y, s, t_1, \dots, t_n)$ AND $\sigma, \tau_1, \dots, \tau_n$

  IF $(\forall x \in \sigma_G \, \exists! y \, \varphi(x, y, \sigma_G, \tau_{1,G}, \dots, \tau_{n,G}))^{M[G]}$

  THEN THERE IS $\varrho \in M^{\mathbb{P}}$ SUCH THAT

  --- $\exists y \in \varrho_G$ ---

  BACK IN M USE REFLECTION: THERE IS A β

  SUCH THAT $\forall \pi \in \operatorname{dom} \sigma \, \forall p \in \mathbb{P}$

  $[\exists \mu \in M^{\mathbb{P}} \, p \Vdash \varphi(\pi, \mu, \sigma, \tau) \rightarrow \exists \mu \in V_\beta \, p \Vdash (\pi, \mu, \sigma, \tau)]$

  TAKE $R = V_\beta \cap M^{\mathbb{P}}$ AND $\varrho = R \times \{\mathbb{1}\}$

  $\varrho_G = \{ \mu_G : \mu \in R \}$ --- THIS WORKS.