SET THEORY
2021-11-29.

$\varphi_1 \ldots\ldots \varphi_k$ AXIOMS

FIND V_β S.T. $\varphi_1^{V_\beta}, \ldots, \varphi_k^{V_\beta}$

$\downarrow$ LÖWENHEIM-SKOLEM

$|A| \leq \aleph_0$ $\varphi_1^A, \ldots, \varphi_k^A$

$\downarrow$ MOSTOWSKI

M CTBLE TRANSITIVE
$\varphi_1^M, \ldots, \varphi_k^M$

EXT, PAIR, UNION, POWERSET
INFINITY, REGULARITY

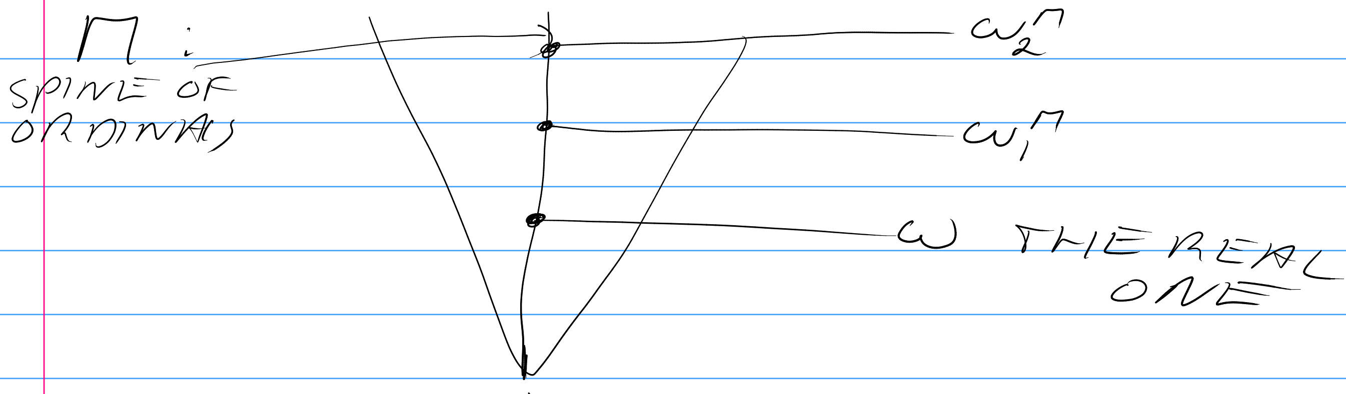
NOW GET M TO SATISFY $\neg CH$

WE EXTEND M TO N
SO THAT $\varphi_1^N, \ldots, \varphi_k^N$ HOLD
<u>AND</u> $(\neg CH)^N$

(WE CHEAT A BIT
SIMPLY ASSUME
M SATISFIES ZFC
AND SHOW THAT N ALSO
SATISFIES ZFC
AND N SATISFIES $\neg CH$

YOU CAN AUDIT THE
PROOF AND "SIMPLY"

CHECK WHICH AXIOMS YOU NEED IN M TO GET $\varphi_1^N, \ldots, \varphi_k^N$ AND $(\neg CH)^N$



M SATISFIES "THERE IS A SMALLEST $\alpha > \omega$ WITH NO BIJECTION TO ω!"
THAT α IS ω_1^M
LIKEWISE THERE MUST BE A β THAT IS ω_2^M

$$\omega_1^M = \min \left\{ \alpha \in M : \underline{\forall f \in M} \begin{array}{l} - f \text{ FUNCTION} \\ - \text{DOM } f = \omega \\ - \alpha \notin \text{RAN } f \end{array} \right\}$$

LAST WEEK GI
- $\mathcal{P}(\omega) \in V_\beta$
- $\mathcal{P}(\omega) \in A$
- $\mathcal{P}(\omega) \notin M$

$\underbrace{\mathcal{P}(\omega) \cap M}_{} \in M$

$$\pi(\mathcal{P}(\omega)) = \{\pi(x) : x \in A \cap \mathcal{P}(\omega)\}$$
$$= \{x : \underline{x \in A \cap \mathcal{P}(\omega)}\}$$
$$= \mathcal{P}(\omega) \cap M$$
$$\llcorner\ \underline{\mathcal{P}^M(\omega)}$$

WE EXTEND M HORIZONTALLY
M AND N WILL HAVE THE SAME ORDINALS.

$\neg$CH: MAKE AN INJECTION f
FROM ω_2^M INTO $\mathcal{P}(\omega)$
(CERTAINLY IF CH HOLDS IN M)

$f \notin M$ f COMES FROM OUTSIDE
ADD IT TO M
EASY ENOUGH: WE SEE MANY
SUCH f

$$N = M[f]$$

TAKE V_β WITH $A \cup \{f\}$
DO LÖWENHEIM-SKOLEM
AND MOSTOWSKI

WHAT COULD POSSIBLY GO WRONG?

- f COULD BE A BIJECTION
  BETWEEN ω_2^M AND $\mathcal{P}^M(\omega)$
  CANTOR BERNSTEIN:
  $(CH)^M$ ALSO BIJECTION $\omega_1^M \leftrightarrow \omega_2^M$
  BUMMER ω_2^M IS NOT ω_2^N
- $f \restriction \omega$ COULD BE A
  BIJECTION $\omega \leftrightarrow \mathcal{P}^M(\omega)$
  REALLY BAD
- $f[\omega_2^M]$ COULD CONTAIN UGLY
  SUBSETS OF ω
  $M \cap On$ IS A COUNTABLE
  ORDINAL δ
  ONE OF THOSE SETS COULD
  CODE A WELL-ORDER OF ω
  IN TYPE δ
  BIJECTION $\omega \leftrightarrow \omega \times \omega$

X —— $X' \leftarrow$ WELL-ORDER
$\wr$

WE NEED TO BUILD THE f WITH CARE (LOTS OF CARE).

HOW? INVOLVE M MUCH MORE

HOW? MANY WAYS. WE USE COHEN'S ORIGINAL METHOD AS (RE) FORMULATED BY SCHOENFIELD

USE FINITE APROXIMATIONS TO f

$\mathbb{P} = Fn(\omega_2 \times \omega, 2)$

IS THE SET OF p WHERE

- p IS A FUNCTION
- $\mathrm{DOM}\, p$ IS A FINITE SUBSET OF $\omega_2 \times \omega$
- $\mathrm{RAN}\, p \subseteq 2 = \{0,1\}$

BETTER WE DO THIS IN M:

$\mathbb{P}^M = \{p \in \mathbb{P} : \mathrm{DOM}\, p \subseteq \omega_2^M \times \omega\}$

!

THIS BELONGS ON PAGE 1

Halle d. 29ten Nov. 73.

Gestatten (*) Sie mir, Ihnen eine Frage vorzulegen, die für mich ein gewisses theoretisches Interesse hat, die ich mir aber nicht beantworten kann; vielleicht können Sie es, und sind so gut, mir darüber zu schreiben, es handelt sich um folgendes.

Man nehme den Inbegriff aller positiven ganzzahligen Individuen n und bezeichne ihn mit (n); ferner denke man sich etwa den Inbegriff aller positiven reellen Zahlgrössen x und bezeichne ihn mit (x); so ist die Frage einfach die, ob sich (n) dem (x) so zuordnen lasse, dass zu jedem Individuum des einen Inbegriffes ein und nur eines des andern gehört? Auf den ersten Anblick sagt man sich, nein es ist nicht möglich, denn (n) besteht aus discreten Theilen, (x) aber bildet ein Continuum; nur ist mit diesem Einwande nichts gewonnen und so sehr ich mich auch zu der Ansicht neige, dass (n) und (x) keine eindeutige Zuordnung gestatten, kann ich doch den Grund nicht finden und um den ist es mir zn thun, vielleicht ist er ein sehr einfacher.

HOW DO WE GET f FROM THIS?

IF $p \in \mathbb{P}$ AND $\langle \alpha, n \rangle \in \operatorname{dom} p$ THEN p TELLS US:

$$\begin{cases} n \in f(\alpha) & \text{IF } p(\alpha, n) = 1 \\ n \notin f(\alpha) & \text{IF } p(\alpha, n) = 0. \end{cases}$$

HOW TO COMBINE THESE FINITE THINGS INTO ONE BIG f?

NOTATION $p \leq q$ MEANS $p \supseteq q$

$\leq$: IS "STRONGER" "p RESTRICTS MORE"

IF p AND q ARE SUCH THAT
$\langle \alpha, n \rangle \in \operatorname{dom} p \cap \operatorname{dom} q$
$p(\alpha, n) = 1$ AND $q(\alpha, n) = 0$
THE WE CANNOT USE THEM <u>BOTH</u>
p AND q ARE INCOMPATIBLE
NOTATION: $p \perp q$
OTHERWISE <u>COMPATIBLE</u>

p AND q COMPATIBLE MEANS $p \cup q$ IS A FUNCTION.

WE BUILD f BY TAKING A SUITABLE SUBSET G OF $\mathbb{P}$

- $\bigcup G$ IS A FUNCTION
- IF $p, q \in G$ THEN $p \cup q \in G$
- $\operatorname{DOM} \bigcup G = \omega_2^M \times \omega$

WE WANT $\boxed{f(\alpha) = \{n : \bigcup G(\alpha, n) = 1\}}$

- IF $\alpha \neq \beta$ THEN $f(\alpha) \neq f(\beta)$

WE WANT SOME n IN $f(\alpha) \triangle f(\beta)$

WE GET THIS: BY <u>$\bigcup G(\alpha, n) \neq \bigcup G(\beta, n)$</u>

WE WANT p IN G THAT DOES THIS

WE WANT $p \in G$ AND $n \in \omega$ SUCH THAT $\langle \alpha, n \rangle, \langle \beta, n \rangle \in p$ AND $p(\alpha, n) \neq p(\beta, n)$

WE WANT G TO BE A FILTER

→ • $\forall p, q \in G \; \exists r \in G \quad r \leq p, q$

• $\forall p \in G \; \forall q \geq p \quad q \in G$

• $\operatorname{DOM} \bigcup G = \omega_2^M \times \omega$

$D_{\alpha, n} = \{ p \in \mathbb{P}^M : \langle \alpha, n \rangle \in p \}$

DEMAND:

$\Rightarrow \forall \alpha \forall n \quad G \cap D_{\alpha, n} \neq \emptyset$

$D^M_{\alpha, n}$ IS DENSE IN $\mathbb{P}^M$

$\forall p \in \mathbb{P} \; \exists q \in D_{\alpha, n} \quad q \leq p$

– TAKE $p \in \mathbb{P}$

• $\langle \alpha, n \rangle \in \operatorname{DOM} p$ DONE!

• $\langle \alpha, n \rangle \notin \operatorname{DOM} p$

$q = p \cup \{ \langle \langle \alpha, n \rangle, 0 \rangle \} \in D_{\alpha, n}$.

$q \leq p$

• f INJECTIVE!

$E_{\alpha, \beta} = \{ p \in \mathbb{P}^M : \exists n \in \omega (\langle \alpha, n \rangle, \langle \beta, n \rangle \in p$ AND $p(\alpha, n) \neq p(\beta, n)) \}$

WE WANT $\forall \alpha, \beta \in \omega_2^M$

$\Rightarrow \alpha \neq \beta \to G \cap E_{\alpha, \beta} \neq \emptyset$

$E_{\alpha, \beta}$ IS DENSE

IF $p \in \mathbb{P}$ PICK n SUCH THAT FOR NO γ DO WE HAVE $\langle \gamma, n \rangle \in \operatorname{DOM} p$

NOW DEFINE $q =$

$p \cup \{ \langle \langle \alpha, n \rangle, 0 \rangle, \langle \langle \beta, n \rangle, 1 \rangle \}$

THEN $q \leq p$ AND $q \in E_{\alpha, \beta}$

REMEMBER: M IS COUNTABLE

$\{D_{\alpha,n} : \alpha \in \omega_2^M, n \in \omega\} \cup \{E_{\alpha,\beta} : \alpha \in \beta \in \omega_2^M\}$
IS A COUNTABLE FAMILY

ENUMERATE THE FAMILY
TAKE SOME $p \in \mathbb{P}$
BUILD A SEQUENCE IN $\mathbb{P}$
$\langle p_n : n \in \omega \rangle$

$$p_0 \leq p \qquad p_0 \in D_0$$
$$p_1 \leq p_0 \qquad p_1 \in D_1$$
$$p_{n+1} \leq p_n \qquad p_{n+1} \in D_{n+1}$$

$$G = \{q \in \mathbb{P} : (\exists n \in \omega)(p_n \leq q)\}$$

EVEN BETTER
WE COULD ENUMERATE
$\mathcal{D} = \{D \in M : D \subseteq \mathbb{P}^M, D \text{ DENSE}\}$
AS $\{D_n : n \in \omega\}$
WE COULD GET G THAT
INTERSECTS ALL OF THEM

SUCH A G IS CALLED M-GENERIC
(ON $\mathbb{P}$)

WE GET G
$\operatorname{DOM} \bigcup G = \omega_2^M \times \omega$
THE f_G THAT WE MAKE OUT OF G
IS AN INJECTION ~~OF~~ OF ω_2^M INTO $\mathcal{P}(\omega)$

$N = M[G]$ IT WILL BE THE SMALLEST MODEL THAT CONTAINS $M \cup \{G\}$

QUESTION
WE HAVE f_G -- $\operatorname{RAN} f_G \in N$

SO $f_G : \omega_2^M \longrightarrow P^N(\omega)$
IS INJECTIVE
WE NEED AN INJECTION
FROM ω_2^N INTO $P^N(\omega)$.
NEXT HOUR:
$\omega_1^M = \omega_1^N$ AND $\omega_2^M = \omega_2^N$
NO MAP $g : \omega \longrightarrow \omega_1^M$ IN N
NO MAP $h : \omega_1^M \longrightarrow \omega_2^M$ IN N
CAN BE SURJECTIVE

$X, Y \in M$ $\quad g : X \longrightarrow Y \quad g \in N$

$g \subseteq X \times Y$
NEXT WEEK WE'LL SEE:
THERE IS A RELATION R
- $R \subseteq \mathbb{P} \times (X \times Y)$ $\quad R \in M$
- $R[G] = \{ \langle x, y \rangle : (\exists p \in G)(\langle p, \langle x, y \rangle \rangle \in R) \}$
$= g$
- FOR EVERY M-GENERIC SET H
$R[H]$ IS A MAP FROM X TO Y.

- Suppose WE HAVE $x \in X$
AND $y_1, y_2 \in Y$ DISTINCT
ASSUME $p_1, p_2 \in \mathbb{P}$ ARE SUCH
THAT $\langle p_1, \langle x, y_1 \rangle \rangle \in R$
$\langle p_2, \langle x, y_2 \rangle \rangle \in R$
THEN p_1 AND p_2 ARE
INCOMPATIBLE: $p_1 \cup p_2$ IS
NOT A FUNCTION.
ASSUME p_1 AND p_2 ARE COMPATIBLE
THEN $p_1 \cup p_2 \in \mathbb{P}$

TAKE AN M-GENERIC H
WITH $p_1 \cup p_2 \in H$.

NOW $\langle x, y_1 \rangle$, $\langle x, y_2 \rangle \in R[H]$

SO $R[H]$ IS NOT A MAP

CONTRADICTION.

TAKE $x \in X$ AND PUT

$$I_x = \{ y \in Y : (\exists p \in \mathbb{P})(\langle p, \langle x, y \rangle \rangle \in R) \}$$

WE KNOW $g(x) \in I_x$

(THERE IS $p \in G$ $\langle p, \langle x, g(x) \rangle \rangle \in R$)

- $I_x \in M$
- IF WE TAKE FOR EACH $y \in I_x$ A $p_y \in \mathbb{P}$ WITH $\langle p_y, \langle x, y \rangle \rangle \in R$ THEN $A_x = \{ p_y : y \in I_x \}$ IS PAIRWISE INCOMPATIBLE

  A_x IS CALLED AN <u>ANTICHAIN</u>
- WE PROVE (IN A MOMENT): EACH A_x IS COUNTABLE HENCE SO IS EACH I_x.

WE KNOW

$$\underbrace{g[X]}_{\text{IN } N} \subseteq \boxed{\underbrace{\bigcup_{x \in X} I_x}_{\text{IN } M}}$$

CARDINALITY IN M

$\leq |X| \cdot \aleph_0$

$N \ni g : \omega \longrightarrow \omega_1^M$

$N \ni h : \omega_1^M \longrightarrow \omega_2^M$

$g[\omega]$ IS COUNTABLE IN N

$h[\omega_1^M]$ IS CONTAINED IN A SET FROM M OF CARD. $\aleph_1^M$

$h[\omega_1^M] \neq \omega_2^M$

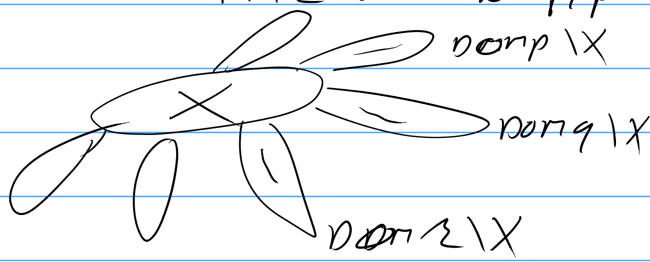
THEOREM

EVERY ANTICHAIN IN $\mathbb{P}$ IS COUNTABLE

LET $A \subseteq \mathbb{P}$ BE UNCOUNTABLE

STEP 1 Δ-SYSTEM LEMMA

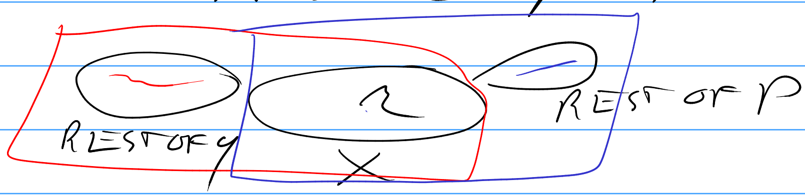
THERE ARE A FIXED FINITE SET X AND $B \subseteq A$ UNCOUNTABLE SUCH THAT IF $p \neq q$ IN B THEN $\operatorname{dom} p \cap \operatorname{dom} q = X$



THERE ARE $2^{|X|}$ FUNCTIONS FROM X TO $\{0,1\}$

THERE IS ONE $r : X \longrightarrow 2$ AND $C \subseteq B$ UNCOUNTABLE SUCH THAT: $p \restriction X = r \quad p \in C$

ALL PAIRS OF ELEMENTS OF C ARE COMPATIBLE



IN N WE DO HAVE A INJECTION $\omega_2 \hookrightarrow P(\omega)$

N SATISFIES $\neg CH$

WHAT IS $2^{\aleph_0}$ IN N?

IT IS $\aleph_2^{\aleph_0}$