SET THEORY 2021-11-22

MODELS, ABSOLUTENESS, AND SOME REFLECTION.

HOW TO PROVE THAT YOU CANNOT PROVE CH ?
[FROM ZFC]

MATH LOGIC
BUILD A MODEL OF ZFC IN WHICH $\neg$CH HOLDS

TWO DIFFICULTIES
1 GÖDEL'S INCOMPLETENESS THM'S

BUILDING A MODEL USES SETS
THAT'S IMPOSSIBLE:
ZFC CANNOT PROVIDE A MODEL FOR ZFC.

A PROOF OF CH (IF ANY) WOULD USE (ONLY) FINITELY MANY AXIOMS. $\varphi_1, \ldots, \varphi_k$

MATH LOGIC SAYS FIND AN INTERPRETATION OF LANG. OF SET TH IN WHICH $\varphi_1, \ldots, \varphi_k$ ARE VALID, TRUE, ...

BUT CH NOT SO $\neg$CH HOLDS.

IT DOES NOT PROVIDE A PROOF IN ZFC THAT CH IS NOT PROVABLE

IT DOES TELL US THAT ANY PROOF OF CH THAT WE MIGHT FIND CANNOT BE TRANSLATED INTO A FORMAL PROOF IN THE FIRST ORDER THEORY ZFC.

FORCING IS A METHOD THAT LETS US DO THIS SYSTEMATICALLY

(2) IT IS ALL QUITE TECHNICAL
- YOU NEED TO KNOW SETS
- PARTIAL ORDERS
- FILTERS, DENSE SETS
- INFINITE COMBINATORICS

M WILL, IN THE END, BE COUNTABLE AND TRANSITIVE.

OUR INTERPRETATION OF $\in$ WILL BE $\in$

LOOK AT SETS, M, WITH $\in$ AS INT. OF $\in$ AND $=$ IS $=$

V_ω MODELS FST

LET M BE A SET
WE CONVERT EVERY FORMULA φ INTO ITS <u>RELATIVATION</u> φ^M TO M
- $(x \in y)^M$ IS $x \in y$
- $(x = y)^M$ IS $x = y$
- $(\neg\varphi)^M$ IS $\neg(\varphi^M)$; $(\varphi \wedge \psi)^M$ IS $(\varphi^M) \wedge (\psi^M)$
- $(\exists x\, \varphi)^M$ IS $(\exists x)(x \in M \wedge \varphi^M)$

$(\exists x \in M)\, \varphi^M$

$(\forall x\, \varphi)^M$ VIA $\neg(\exists x)(\neg\varphi)$ -- $(\forall x \in M)\, \varphi^M$

WE SAY "φ IS TRUE/VALID IN M"
OR "M SATISFIES φ"
IF φ^M IS PROVABLE (IN ZFC)

$(\text{EXTENSIONALITY})^M$ IS

$$\forall x\in M\ \forall y\in M\ \big(\underbrace{\forall z\in M\,(z\in x \Leftrightarrow z\in y)}_{x\cap M = y\cap M} \rightarrow x=y\big)$$

SO IF M IS TRANSITIVE
THEN M SATISFIES EXTENSIONALITY
BECAUSE $x\cap M = x$, IF $x\in M$

! WE ASSUME THAT
EXT, PAIRING, UNION, POWER SET
INFINITY, REGULARITY, AND CHOICE
ARE IN OUR LIST.

THE ULTIMATE GOAL:
FIND M SUCH THAT
$\varphi_1^M, \ldots, \varphi_k^M$ HOLD FIRST
AND $(\neg CH)^M$ ALSO LATER

WE ASSUME ZF(C) IS CONSISTENT
AND THAT ALL AXIOMS HOLD
IN $V = \bigcup_\alpha V_\alpha$
FIRST STEP FIND V_α SUCH THAT
$\varphi_1^{V_\alpha}, \ldots, \varphi_k^{V_\alpha}$ HOLD

ABSOLUTENESS
φ A FORMULA
$\varphi(x_1, \ldots, x_k)$
ITS FREE VARIABLES ARE AMONG
$x_1, \ldots, x_k$

M AND N SETS / CLASSES (V)

IF $M \subseteq N$ THEN φ IS ABSOLUTE FOR M, N.
BETWEEN M AND N

IF:

$$ZFC \vdash \forall m_1, \dots, m_k \in M \left(\varphi^M(m_1, \dots, m_k) \iff \varphi^N(m_1, \dots, m_k) \right)$$

[$\mathbb{Q} \subseteq \mathbb{R}$ $\quad \varphi(x)$ IS $(\exists y)(y \cdot y = x)$
$\varphi^{\mathbb{Q}}(2)$ IS FALSE
$\varphi^{\mathbb{R}}(2)$ IS TRUE]

φ IS ABSOLUTE FOR M
MEANS φ IS ABSOLUTE FOR M, V
OR $\forall m_1 \dots m_k \in M \left(\varphi^M(m_1, \dots, m_k) \iff \varphi(m_1, \dots, m_k) \right)$

JECH: CHAPTER 12 (13) GÖDEL OPERATIONS
KUNEN (1980): CHAPTER IV
CHAPTER V

- IF φ AND ψ ARE ABSOLUTE FOR M, N
THEN SO ARE $\neg\varphi$ AND $\varphi \wedge \psi$.
- ABSOLUTENESS FOR FORMULAS WITHOUT QUANTIFIERS

- QUANTIFIERS?
$\forall x \forall y \exists z \forall w (w \in z \iff (w = x \vee w = y))$
$x \subseteq y \quad (\forall z)(z \in x \to z \in y)$

$(x \subseteq y)^M$ IFF $x \subseteq y$??

$M = \{\emptyset, a\} \qquad a = \{\{\emptyset\}\}$
$(a \subseteq \emptyset)^M \qquad a \not\subseteq \emptyset$

ASSUME M AND N ARE TRANSITIVE
φ IS ABSOLUTE FOR M, N
THEN

$$(\exists x \in y)\varphi \quad ((\exists x)(x \in y \wedge \varphi))$$

IS ABSOLUTE FOR M, N.

$\varphi(x, y, z_1, \ldots, z_m)$

$[(\exists x)(x \in y \wedge \varphi(x, y, z_1, \ldots, z_m))]^M$

$(\exists x \in M)(x \in y \wedge \varphi(x, y, z_1, \ldots, z_m))^M$

$(\exists x \in M)(x \in y \wedge \varphi^M(x, y, z_1, \ldots, z_m))$

ASSUME $y, m_1, \ldots, m_m \in M$

BY TRANSITIVITY $(\exists x \in M)(x \in y \wedge \ldots$

IS $(\exists x \in y)(\ldots)$

1 $(\exists x)(x \in y \wedge \varphi^M(x, y, m_1, \ldots, m_m))$

BUT IF $x \in y$ THEN

$\varphi^M(x, y, m_1, \ldots, m_m) \Leftrightarrow \varphi^N(x, y, m_1, \ldots, m_m)$

2 $(\exists x)(x \in y \wedge \varphi^N(x, y, m_1, \ldots, m_m))$

BY TRANS 3 $[(\exists x)(x \in y \wedge \varphi(x, y, m_1, \ldots, m_m))]^N$

$(\exists x \in N)(x \in y \wedge \varphi^N(x, y, m_1, \ldots, m_m))$

$\exists x \in y$: BOUNDED QUANTIFIER

Δ_0 – FORMULAS ALL QUANTIFIERS BOUNDED

- $x \in y$, $x = y$
- φ, ψ Δ_0 $\rightarrow$ $\neg\varphi$, $\varphi \wedge \psi$ Δ_0
- φ Δ_0 $\rightarrow$ $(\exists x \in y)\varphi$ AND $(\forall x \in y)\varphi$ ALSO Δ_0

IF $M \subseteq N$ AND M AND N ARE TRANSITIVE THEN ALL Δ_0-FORM'S ARE ABSOLUTE FOR M, N.

JECH 12.10 MANY THINGS ARE Δ_0

$x \subseteq y \quad x = y \quad z = \{x, y\} \quad z = \langle x, y \rangle$

$z = \emptyset$, z IS TRANSITIVE

z IS AN ORDINAL ← NEEDS REGULARITY

"x IS UNCOUNTABLE" IS NOT Δ_0
NOT ALWAYS ABSOLUTE
FOR TRANSITIVE SETS.

LET $\varphi_1, \dots, \varphi_m$ BE A LIST OF FORMULAS AND EVERY SUBFORMULA OF EVERY φ_i IS ALSO IN THE LIST.
EQUIVALENT ARE

a) ALL φ_i ARE ABSOLUTE FOR M, N

b) FOR ALL i

IF φ_i IS $(\exists x)\, \varphi_j(x, y_1, \dots, y_k)$
THEN
$\forall m_1, \dots, m_k \in M$
$[(\exists x \in N)\, \varphi_j^N(x, m_1, \dots, m_k) \longrightarrow (\exists x \in M)\, \varphi_j^N(x, m_1, \dots, m_k)]$

a) ⇒ b) LET $m_1, \dots, m_k \in M$

ASSUME $(\exists x \in N)\, \varphi_j^N(x, m_1, \dots, m_k) \longleftarrow \varphi_i^N$

THEN $\varphi_i^N(m_1, \dots, m_k)$

SO $\varphi_i^M(m_1, \dots, m_k)$ HOLDS

OR $(\exists x \in M)\, \varphi_j^M(x, m_1, \dots, m_k)$ HOLDS

ABS. OF φ_j : $(\exists x \in M)\, \varphi_j^N(x, m_1, \dots, m_k)$

TARSKI–VAUGHT

b) $\Longrightarrow$ a) INDUCTION ON COMPLEXITY

ATOMIC $\quad x \in y \quad x = y$ CLEAR

$\varphi_i = \varphi_j \wedge \varphi_k \qquad \varphi_i = \neg\varphi_j$ CLEAR

ONLY PROBLEM

$\varphi_i(y_1, \dots, y_\ell)$ IS $(\exists x)\, \varphi_j(x, y_1, \dots, y_\ell)$

LET $y_1, \dots, y_\ell \in M$

$$\begin{aligned}
\varphi_i^M(y_1, \dots, y_\ell) &\overset{\text{DEF}}{\Longleftrightarrow} (\exists x \in M)\, \varphi_j^M(x, y_1, \dots, y_\ell) \\
\text{IND. ASS} \quad &\Longleftrightarrow (\exists x \in M)\, \varphi_j^N(x, y_1, \dots, y_\ell) \\
\text{ASSUMPTION b} \quad &\Longleftrightarrow (\exists x \in N)\, \varphi_j^N(x, y_1, \dots, y_\ell) \\
&\Longleftrightarrow \varphi_i^N(y_1, \dots, y_\ell)
\end{aligned}$$

REFLECTION

LOOK AT OUR FORMULAS $\varphi_1, \dots, \varphi_k$

OUR LIST IS SUBFORM. CLOSED

THEN

$\forall \alpha\ \exists \beta > \alpha$ "$\varphi_1, \dots, \varphi_k$ ARE ABSOLUTE FOR V_β"

SO FOR OUR AXIOMS:

$$(\forall \alpha)(\exists \beta > \alpha)\left[\varphi_1^{V_\beta} \wedge \dots \wedge \varphi_k^{V_\beta}\right]$$

$N = V \qquad \varphi_i^N$ IS JUST φ_i

IF φ_i IS $(\exists x)\, \varphi_j(x, y_1, \dots, y_\ell)$

DEFINE

$$G_i(y_1, \dots, y_\ell) = \begin{cases} 0 & \text{IF } \neg\exists x\, \varphi_j(x, y_1, \dots, y_\ell) \\ \min\{\eta : \exists x \in V_\eta\, \varphi_j(x, y_1, \dots, y_\ell)\} & \text{IF } \exists x\, \varphi_j(x, y_1, \dots, y_\ell) \end{cases}$$

$$F_i(\zeta) = \sup\{G(y_1, \dots, y_\ell) : (y_1, \dots, y_\ell) \in V_\zeta^\ell\}$$

α GIVEN: $\quad \beta_0 = \alpha$

GIVEN $\beta_p \qquad \beta_{p+1} = \max\{\beta_p + 1, F_1(\beta_p), \dots, F_k(\beta_p)\}$

TAKE $\quad \beta = \sup_{p \in \omega} \beta_p$

APPLY TARSKI-VAUGHT TO V_β AND V

BECAUSE WE KNOW

FOR $y_1, \dots, y_\ell \in V_\beta$

IF $\exists x \in V\ \varphi_j(x, y_1, \dots, y_\ell)$

THEN $\exists x \in V_\beta\ \varphi_j(x, y_1, \dots, y_\ell)$

$y_1, \dots, y_\ell \in V_{\beta_p}$

$x \in V_{\beta_{p+1}}$

A COUNTABLE M?

APPLY AC

WELLORDER V_β USING $\lhd$

DEFINE IF $\varphi_i = (\exists x)\ \varphi_j(x, y_1, \dots, y_{\ell_i})$

$$H_i : V_\beta^{\ell_i} \longrightarrow V_\beta$$

$$(y_1, \dots, y_\ell) \longmapsto \begin{cases} 0 & \text{IF } \neg \exists x\, \varphi_j \\ \lhd\text{-}\min\{x : \varphi_j(x, y_1, \dots, y_\ell)\} & \text{IF } \exists x\, \varphi_j \end{cases}$$

TAKE $X \subseteq V_\beta$ (SMALL)

$A_0 = X$

$A_{n+1} = A_n \cup \bigcup_{i=1}^{k} H_i[A^{\ell_i}]$

$A = \bigcup_{n \in \omega} A_n$

- $H_i[A^{\ell_i}] \subseteq A$
- $|A| \leq \max\{|X|, \aleph_0\}$
- ALL FORMULAS ARE ABSOLUTE FOR A, V_β

JECH 6.15

WE CAN DEFINE $\pi : A \longrightarrow V$

$$\pi(x) = \{\pi(y) : y \in A \cap x\}$$

[ACTUALLY BECAUSE WE CAN ADD A FEW AXIOMS WE CAN HAVE $V_\omega \subseteq A$]

REPLACEMENT $M = \pi[A]$ IS A SET

EXTENSIONALITY: $\pi : A \longrightarrow M$ IS BIJECTIVE AND AN $\in$-ISOMORPHISM

FOR ALL THE AXIOMS IN OUR LIST

$\varphi_i^M \iff \varphi_i^A \iff \varphi_i^{V_\beta} \iff \varphi_i$

TRUE! TRUE!

SKOLEM'S PARADOX

IN M : $\mathbb{R}$ EXISTS AND IS UNCOUNTABLE

??????

=

$\varphi_1, \dots, \varphi_k$

V_β ST $\varphi_1, \dots, \varphi_k$ HOLD IN V_β

$A \subseteq V_\beta$ COUNTABLE $\varphi_1 - \varphi_k$ HOLD IN A

M MOSTOWSKI-COLLAPSE OF A
ISOMORPHIC TO A. $\varphi_1 - \varphi_k$ HOLD IN M.

NEXT WEEK
TAKE SUCH AN M AND
EXTEND IT TO AN N
IN WHICH OUR φ_i'S STILL HOLD
AND ALSO $\underline{\neg CH}$