SET THEORY 2021-11-15

QUESTION 1936 RUZIEWICZ
LET κ BE A CARDINAL,
$\lambda < \kappa$ ANOTHER CARDINAL
LET $F : \kappa \longrightarrow [\kappa]^{<\lambda}$ BE
A MAP SUCH THAT $\alpha \notin F(\alpha)$ (ALL α)
MUST THERE BE A FREE SUBSET
OF CARDINALITY κ FOR F

S IS FREE : $S \cap \bigcup_{\alpha \in S} F(\alpha) = \emptyset$

OTHER FORMULATION

$\alpha, \beta \in S \quad \alpha \neq \beta : \quad \alpha \notin F(\beta)$

HOMEWORK : κ REGULAR

- WE KNOW $\sup F(\alpha) < \kappa$

THERE IS A CUB SET C
SUCH THAT
$\forall \delta \in C \ \forall \alpha < \delta \quad \sup F(\alpha) < \delta$
IF $\alpha < \beta$ IN C THEN $\beta \notin F(\alpha)$
($\alpha \notin F(\beta)$ OR $\alpha \in F(\beta)$???)

- λ REGULAR

$f : E^{\kappa}_{\lambda} \longrightarrow \kappa \qquad f(\alpha) = \sup(\alpha \cap F(\alpha)) < \alpha$
TAKE T STATIONARY AND $\gamma < \kappa$
SUCH THAT $f(\alpha) = \gamma \qquad (\alpha \in T)$

- LOOK AT $S = C \cap T$
  - S STATIONARY
  - IF $\alpha < \beta$ IN S : $\quad \beta \notin F(\alpha)$

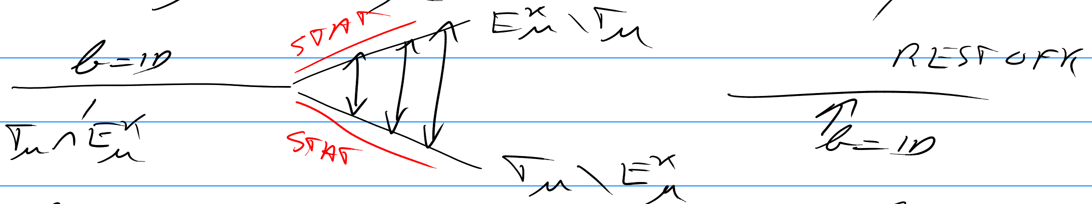
$\alpha \in (\gamma, \beta)$ AND $(\gamma, \beta) \cap F(\beta) = \emptyset$
SO $\alpha \notin F(\beta)$

λ IS SINGULAR

$\kappa = \bigcup_{\substack{\mu < \lambda \\ \mu \text{ CARDINAL}}} T_\mu \quad : \ T_\mu = \{\alpha : |F(\alpha)| < \mu\}$

SO SOME T_μ IS STATIONARY

IF $T_\mu \cap E^\kappa_\mu$ IS STATIONARY: DONE!



$b : \kappa \to \kappa$ BIJECTION $b \circ b = ID$

$G : \kappa \to [\kappa]^{<\lambda}$

$$G(\alpha) = b[F(b(\alpha))]$$

FIND $S \subseteq E^\kappa_\mu$ FREE CARD. κ

$$S \cap \bigcup\{G(\alpha) : \alpha \in S\} = \emptyset$$

$$S \cap \bigcup\{b[F(b(\alpha))] : \alpha \in S\} = \emptyset$$

$$S \cap b[\bigcup\{F(b(\alpha)) : \alpha \in S\}] = \emptyset$$

$$b[S] \cap \bigcup\{F(b(\alpha)) : \alpha \in S\} = \emptyset$$

$b[S]$ IS FREE FOR F.

SINGULAR CASE: 1961 HAJNAL

WE HAVE

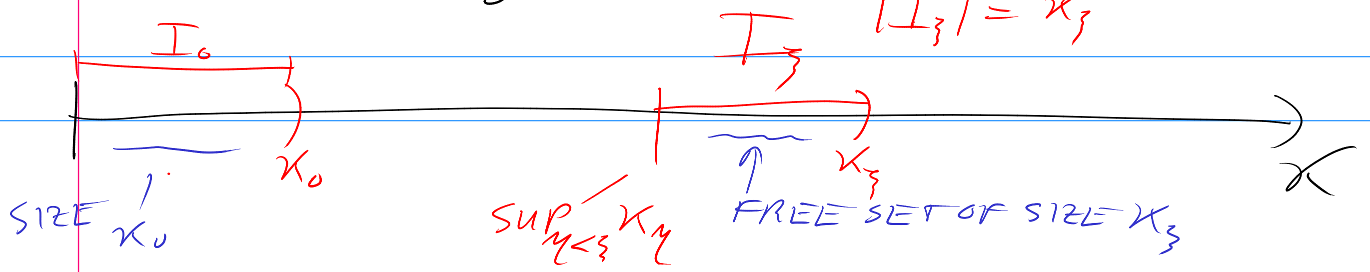
- κ SINGULAR
- $\lambda < \kappa$
- $F : \kappa \to [\kappa]^{<\lambda}$

LET $\langle \kappa_\zeta : \zeta < cf\,\kappa \rangle$ BE AN INCREASING AND COFINAL SEQUENCE OF REGULAR CARDINALS IN κ. (SUCCESSORS EVEN)

WE CAN ASSUME

$cf\,\kappa < \lambda$ [IF WE CAN PROVE IT FOR LARGER λ ...]

$\lambda^+ < \kappa_0$



- A SUBSTITUTE FOR THE CUB SET C
  ($\delta \in C \wedge \alpha < \delta \rightarrow \sup F(\alpha) < \delta$)

WE MAKE $\langle A_\xi : \xi < \mathrm{cf}\,\kappa \rangle$

- $\eta < \xi \rightarrow A_\eta \subseteq A_\xi$
  $\bigcup_{\xi < \mathrm{cf}\,\kappa} A_\xi = \kappa \qquad (I_\xi \subseteq A_\xi)$
- $|A_\xi| = \kappa_\xi$
- IF $\alpha \in A_\xi$ THEN $F(\alpha) \subseteq A_\xi$

IF WE HAVE $\langle A_\eta : \eta < \xi \rangle$

THEN $A_{\xi,0} = I_\xi \cup \bigcup_{\eta<\xi} A_\eta \qquad |A_{\xi,0}| = \kappa_\xi$

$A_{\xi,n+1} = \bigcup \{ F(\alpha) : \alpha \in A_{\xi,n} \}$

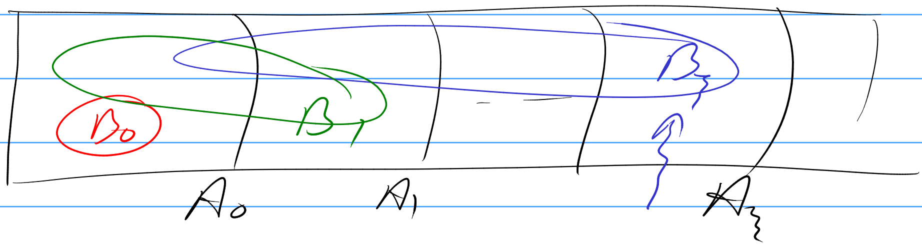
$$A_\xi = \bigcup_{n \in \omega} A_{\xi,n}$$

$$|A_\xi| = \kappa_\xi \cdot \aleph_0 = \kappa_\xi$$

$$|A_{\xi,n+1}| \leq \kappa_\xi \cdot \lambda = \kappa_\xi$$

- NOTE IF $\alpha \in A_\xi$ AND $\beta \notin A_\xi$
  THEN $\beta \notin F(\alpha)$

- FIND $\langle B_\xi : \xi < \mathrm{cf}\,\kappa \rangle$
  - $B_\xi \subseteq A_\xi$ AND $|B_\xi| = \kappa_\xi$
  - B_ξ IS FREE FOR F.



$$B_\xi \cap \bigcup \{ F(\alpha) : \alpha \in B_\xi \} = \emptyset$$

WE MAKE B_ξ SMALLER

WE MAKE $\langle C_\xi : \xi < cf\kappa \rangle$

— $C_\xi \subseteq B_\xi$; $|C_\xi| = |B_\xi| = \kappa_\xi$

AND

IF $\eta \leq \xi$ AND $\alpha \in C_\eta$ THEN
$$F(\alpha) \cap C_\xi = \emptyset$$

ALREADY TRUE IF $\eta = \xi$

NEEDS WORK FOR $\eta < \xi$

GIVEN $\langle C_\eta : \eta < \xi \rangle$

$$H_\xi = \bigcup_{\eta<\xi} C_\eta \quad ; \quad |H_\xi| \leq \sum_{\eta<\xi} \kappa_\eta < \kappa_\xi$$

$$\left| \bigcup \{F(\alpha) : \alpha \in H_\xi\} \right| \leq |H_\xi| \cdot \lambda < \kappa_\xi$$

JUST TAKE
$$C_\xi = B_\xi \setminus \bigcup \{F(\alpha) : \alpha \in H_\xi\}$$

NOW WE REALLY HAVE THE SUBSTITUTE FOR C

NOW WE HAVE TO LOOK DOWN

STILL WE CAN HAVE

$\eta < \xi$ AND $\alpha \in C_\xi$

WITH $C_\eta \cap F(\alpha) \neq \emptyset$

HERE COMES THE CLEVER BIT.

ONE MORE SEQUENCE OF SETS

$\langle D_\xi : \xi < cf\kappa \rangle$

— $D_\xi \subseteq C_\xi$; $|D_\xi| = \kappa_\xi$

— $D_\xi = \bigcup_{\gamma<\lambda^+} D_{\xi,\gamma}$; $D_{\xi,\gamma} \cap D_{\xi,\delta} = \emptyset$ $\gamma \neq \delta$

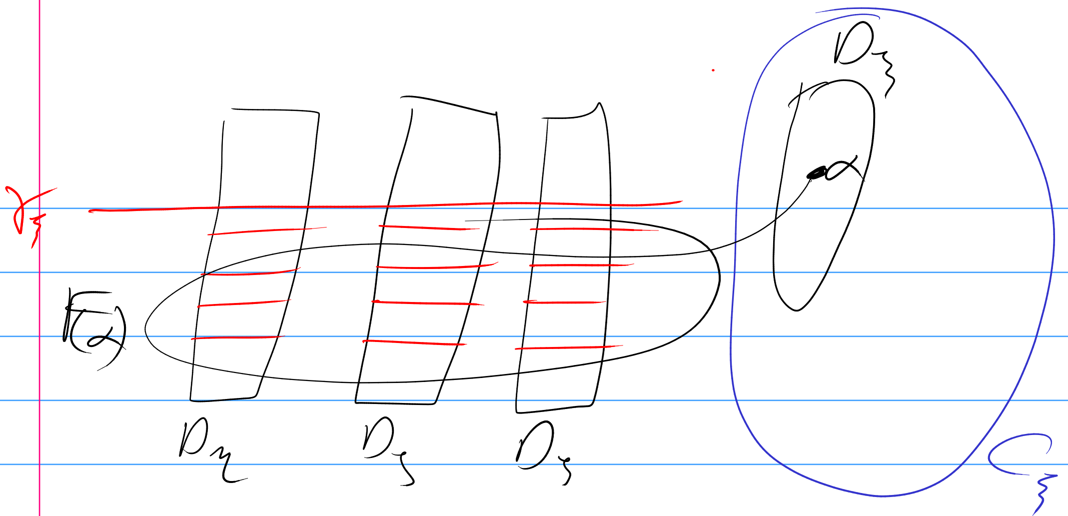
$|D_{\xi,\gamma}| = \kappa_\xi$

$(\kappa_\xi \cdot \lambda^+ = \kappa_\xi)$

— FOR EVERY $\xi < cf\kappa$ THERE IS $\gamma_\xi < \lambda^+$ SUCH THAT

FOR ALL $\alpha \in D_\xi$ AND ALL $\eta < \xi$

IF $\gamma_\xi < \gamma < \lambda^+$ THEN $F(\alpha) \cap D_{\eta,\gamma} = \emptyset$



GIVEN $\langle D_\eta : \eta < \xi \rangle$ $\langle D_{\eta,\gamma} : \eta < \xi, \gamma < \lambda^+ \rangle$

IF $\alpha \in C_\xi$ THEN $|F(\alpha)| < \lambda$

THAT MEANS

IF $\eta < \xi$ THEN THERE IS $\gamma_{\alpha,\eta} < \lambda^+$

SUCH THAT

$$F(\alpha) \cap \bigcup_{\gamma > \gamma_{\alpha,\eta}} D_{\eta,\gamma} = \emptyset$$

WE KNOW: $\xi < \lambda^+$

I HAVE $|\xi|$ MANY $\gamma_{\alpha,\eta}$'S

THERE IS ONE $\gamma_\alpha < \lambda^+$ ABOVE ALL THE $\gamma_{\alpha,\eta}$'S

BUT $\lambda^+ < \kappa_\xi$

$C_\xi = \bigcup_{\gamma < \lambda^+} \{\alpha : \gamma_\alpha \leq \gamma\}$

PICK ONE $\gamma_\xi < \lambda^+$ SUCH THAT

$D_\xi = \{\alpha \in C_\xi : \gamma_\alpha \leq \gamma_\xi\}$

HAS CARDINALITY κ_ξ

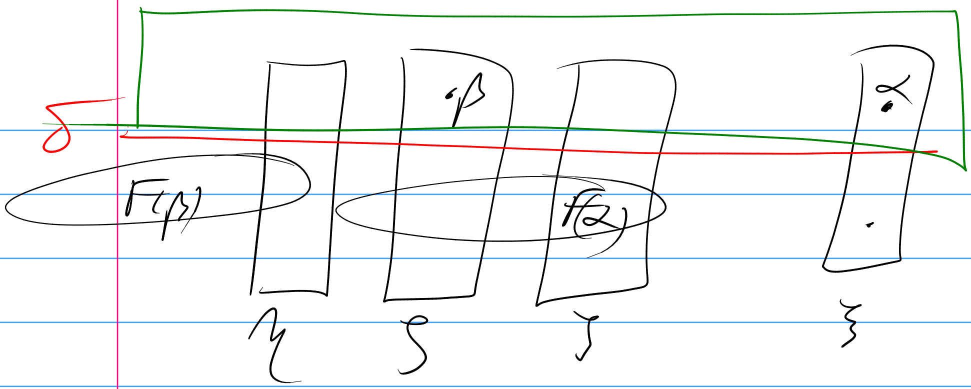
JUST PARTITION D_ξ INTO $\langle D_{\xi,\gamma} : \gamma < \lambda^+ \rangle$

LOOK AT $\langle \gamma_\xi : \xi < cf\,\kappa \rangle$

BUT $cf\,\kappa < \lambda^+$

SO THERE IS ONE $\delta < \lambda^+$

SUCH THAT $\gamma_\xi < \delta$ FOR ALL $\xi < cf\,\kappa$



$$E_\xi = \bigcup_{\gamma>\delta} D_{\xi,\gamma}$$

$$E = \bigcup_{\xi<cf\kappa} E_\xi$$

LET $\alpha, \beta \in E$

$\alpha, \beta \in E_\xi \subseteq D_\xi \subseteq C_\xi \subseteq B_\xi$: $\alpha \notin F(\beta)$ $\beta \notin F(\alpha)$

$\alpha \in E_\eta$ $\beta \in E_\xi$ $\eta < \xi$

$\beta \notin F(\alpha)$ BECAUSE $\beta \in C_\xi$

$\alpha \notin F(\beta)$ BECAUSE $F(\beta) \cap E_\eta = \emptyset$

E IS FREE AND $|E| \geq \sup_{\xi<cf\kappa} \kappa_\xi = \kappa$.

PARTITION CALCULUS

THERE IS NO COMPLETE DISORDER

IF YOU COVER AN INFINITE SET WITH FINITELY MANY SETS (AT LEAST) ONE OF THE SETS IS INFINITE.

GENERALISE THIS

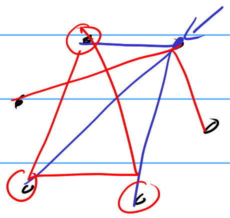
RAMSEY'S THEOREM

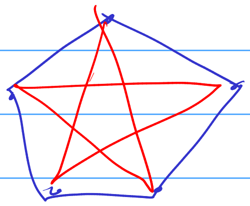
IF $n, k \in \omega$ AND $F : [\omega]^n \to k$ THEN THERE IS AN INFINITE SET H SUCH THAT F IS CONSTANT ON $[H]^n$.

$n = 1$: $F : \omega \to k$ IS CONSTANT ON AN INFINITE SET

$n = 2$: $[\omega]^2$ IS THE COMPLETE GRAPH ON ω.
$F : [\omega]^2 \to k$ REPRESENTS A COLOURING.
THERE IS AN INFINITE SET H SUCH THAT ALL EDGES BETWEEN MEMBERS OF H HAVE THE SAME COLOUR

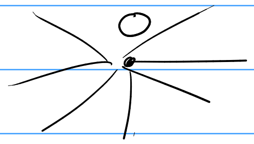
$F : [6]^2 \to 2$ IS CONSTANT ON A TRIANGLE





$n = 1$ DONE

$n = 2$



$[\omega \setminus \{0\}]^n \to k$

$x \mapsto F(\{0\} \cup x)$

$n \to n+1$

$n \mapsto F(\{0, n\})$ THERE IS $i_0 < k$ AND I_0 INFINITE SUCH THAT $F(\{0, n\}) = i_0$ $n \in I_0$

$F(\{0\} \cup x) = i_0$ $x \in [I_0]^n$

$a_1 = \min I_0$

$n \mapsto F(\{a_1, n\})$ ON $I_0 \setminus \{a_1\}$

$x \mapsto F(\{a_1\} \cup x)$ ON $[I_0 \setminus \{a_1\}]^n$

FIND i AND I_1 INFINITE
$I_1 \subseteq I_0 \setminus \{a_1\}$
$F(\{a_1, m\})$ CONSTANT ON $[I_1]^n$ $F(\{a_1\} \cup x)$

$a_{m+1} = \min I_m$ $\quad$ $a_{m+1} = \min I_m$
$m \mapsto F(\{a_{m+1}, m\})$ ON $I_m \setminus \{a_{m+1}\}$ $\quad$ $F(\{a_{m+1}\} \cup x)$ ON $[I_m \setminus \{a_{m+1}\}]^n$
CONSTANT ON I_{m+1} WITH VALUE i_{m+1}

WE GET $\langle a_n : n \in \omega \rangle$
$\langle i_n : n \in \omega \rangle$
$m_0 < m_1 < \dots < m_n$
$F(\{a_{m_0}, a_{m_1}, \dots, a_{m_n}\}) = i_{m_0}$
IF $n < m$ THEN
$F(\{a_n, a_m\}) = i_n \Leftarrow$
TAKE $I \subseteq \omega$ INFINITE
AND $i < k$ SUCH THAT
$i_m = i$ $m \in I$
$H = \{a_n : n \in I\}$
~~$F(\{a_n, a_m\})$~~ $= i$ FOR ALL $n, m \in I$.
$=$ $F(x) = i$ FOR ALL $x \in [H]^{n+1}$

NOTATION $\aleph_0 \longrightarrow (\aleph_0)^n_k$ $n \leftarrow$ CARD OF SUBS, $k \leftarrow$ NUMBER OF COLOURS
STARTING CARDINALITY $\quad$ TARGET CARDINALITY

FINITE: $6 \longrightarrow (3)^2_2$
$5 \not\longrightarrow (3)^2_2$

MANY QUESTIONS!
$\kappa \longrightarrow (\lambda)^n_k$??

$2^{\aleph_0} \not\longrightarrow (\aleph_1)^2_2$ [SIERPIŃSKI]
$\lhd$ A WELL ORDER OF $\mathbb{R}$
$<$ NORMAL ORDER OF $\mathbb{R}$

$$F(\{x,y\}) = \begin{cases} 0 & x \triangleleft y \iff x<y \\ 1 & x \triangleleft y \iff y<x \end{cases}$$

HOMOGENEOUS: WELL ORDERED BY $<$
OR BY $>$

EXERCISE: IT WILL BE COUNTABLE

$2^{\aleph_0} \not\to (3)^2_{\aleph_0}$ [HOMEWORK]

ERDŐS–RADO

$$(2^{\aleph_0})^+ \longrightarrow (\aleph_1)^2_{\aleph_0}$$

$$\left(\begin{array}{l} 2^\kappa \not\longrightarrow (\kappa^+)^2_2 \\ 2^\kappa \not\longrightarrow (3)^2_\kappa \\ (2^\kappa)^+ \longrightarrow (\kappa^+)^2_\kappa \end{array} \right)$$

JECH'S BOOK

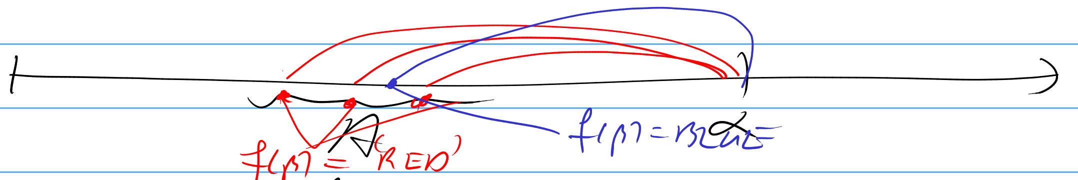
$\lambda = (2^\kappa)^+$

$F : [\lambda]^2 \longrightarrow \kappa$ COLOURING.

SUPPOSE $A \subseteq \lambda$ WITH $|A| \leq \kappa$
AND $f : A \longrightarrow \kappa$

<u>IF</u> THERE IS AN ORDINAL α
SUCH THAT

$$A \subseteq \alpha$$



AND $f(\beta) = F(\{\beta,\alpha\})$ FOR ALL $\beta \in A$

THEN $G(A,f) = \min\{\alpha : f$ GOES WITH $\alpha\}$

IF <u>NOT</u> $G(A,f) = 0$

NEXT TAKE $\alpha < \lambda$

$$|[\alpha]^{\le\kappa}| \le (2^\kappa)^{\le\kappa} \le 2^\kappa$$

EACH $A \in [\alpha]^{\le\kappa}$ HAS $\le \kappa^\kappa = 2^\kappa$ MANY FUNCTIONS TO κ

SO $\{G(A,f)\cancel{+1} : A \in [\alpha]^{\le\kappa}; f : A \to \kappa\}$
IS BOUNDED IN λ
IT HAS CARD $\le 2^\kappa$
LET $g(\alpha)$ BE ITS SUPREMUM

$C = \{\delta : (\forall\alpha<\delta)(g(\alpha)<\delta)\}$ IS CUB

TAKE δ IN C OF COFINALITY κ^+ (AT LEAST)

TAKE $\alpha_0 = 0$
$F(\{\alpha_0, \delta\})$ —— $A = \{\alpha_0\}$, $f(\alpha_0) = F(\{\alpha_0, \delta\})$
LET $\alpha_1 = G(\{\alpha_0\}, f)$
NEXT STEP
$A = \{\alpha_0, \alpha_1\}$ $f(\alpha_0) = F(\{\alpha_0, \delta\})$
$f(\alpha_1) = F(\{\alpha_1, \delta\})$
$\alpha_2 = G(A, f)$ ...
ASSUME WE HAVE $\xi < \kappa^+$
AND $\langle \alpha_\eta : \eta < \xi \rangle$
LOOK AT $A = \{\alpha_\eta : \eta < \xi\}$ $\le \sup_{\eta<\xi} \alpha_\eta + 1 < \delta$
$f(\alpha_\eta) = F(\{\alpha_\eta, \delta\})$
$\alpha_\xi = G(A, f) < \delta$

WE GET $\langle \alpha_\xi : \xi < \kappa^+ \rangle$
IF $\eta < \xi < \kappa^+$
THEN $F(\{\alpha_\eta, \alpha_\xi\}) = F(\{\alpha_\eta, \delta\})$
κ COLOURS κ^+ POINTS

WE FIND $S \subseteq \kappa^+$ AND $\gamma < \kappa$
SUCH THAT $F(\{\alpha_\eta, \delta\}) = \gamma$
FOR ALL $\eta \in S$

BUT THEN (AUTOMATICALLY)

$$F(\{\alpha_\eta, \alpha_\xi\}) = \gamma$$

IF $\eta < \xi$ IN S

$\{\alpha_\xi : \xi \in S\} \cup \{\delta\}$
IS HOMOGENEOUS FOR F