SET THEORY 2021-11-1

CARDINAL ARITHMETIC

[WE ASSUME AC FROM NOW ON]

- WHAT WE KNOW.

  FOR INFINITE CARDINALS κ AND λ WE HAVE
  $$\kappa+\lambda=\kappa\cdot\lambda=\max\{\kappa,\lambda\}$$
  IN TERMS OF ALEPHS:
  $$\aleph_\alpha+\aleph_\beta=\aleph_\alpha\cdot\aleph_\beta=\aleph_{\max\{\alpha,\beta\}}$$
  SO, WE HAVE AN EASY FUNCTION OF PAIRS OF ORDINALS THAT TELLS US THE OUTCOME OF ADDITION AND MULTIPLICATION.

- WHAT WE DO NOT KNOW.

  WHAT ABOUT κ^λ ?

  WE ASSUME AC SO $\kappa^\lambda=\aleph_\gamma$ FOR SOME γ

  HOW DOES γ IN
  $$\aleph_\gamma=\aleph_\alpha^{\aleph_\beta}$$
  DEPEND ON α? AND β?

  SPECIAL CASE: $\alpha=\beta$

  WE ALREADY KNOW

  IF $2\leq\kappa\leq 2^\lambda$

  THEN $2^\lambda\leq\kappa^\lambda\leq(2^\lambda)^\lambda=2^\lambda$

  SO $\kappa^\lambda=2^\lambda$

  AND $\lambda^\lambda=2^\lambda$

  SO: $2^{\aleph_\alpha}=\aleph_\alpha^{\aleph_\alpha}=\aleph_{???}$

- WE SHALL SEE WHAT WE CAN SAY ABOUT THE BEHAVIOUR OF THE CONTINUUM FUNCTION $\kappa\mapsto 2^\kappa$ AND THE EXPONENTIAL FUNCTION $(\kappa,\lambda)\mapsto\kappa^\lambda$ FOR INFINITE κ AND λ.

BUT FIRST A BIT MORE ABOUT COFINALITY.

REMEMBER

$cf(\kappa) = \min\{|C| : C \subseteq \kappa \text{ IS COFINAL}\}.$

CHARACTERIZATIONS

$cf(\kappa) = \min\{\gamma :$ THERE IS AN INCREASING COFINAL $f : \gamma \to \kappa\}$

$cf(\kappa) = \min\{\lambda :$ THERE IS A FAMILY OF SETS $\{S_\alpha : \alpha < \lambda\}$ SUCH THAT $|S_\alpha| < \kappa$ FOR ALL α AND $\kappa = \bigcup_{\alpha<\lambda} S_\alpha\}$.

[SEE GROUP INTERACTION]

PROPERTIES

- $cf(cf(\kappa)) = cf(\kappa)$
- $cf(\kappa)$ IS REGULAR
- SO κ IS REGULAR IFF $\kappa = cf(\kappa)$.
  AND SINGULAR IFF $\kappa > cf(\kappa)$.

So
- $2 \leq \kappa \leq 2^\lambda$ IMPLIES $\kappa^\lambda = 2^\lambda$
  SO WE ONLY NEED TO LOOK AT 2^λ
- MORE INTERESTING:
  WHAT IF $2^\lambda < \kappa$?
  - CONSTANT FUNCTIONS: $\kappa^\lambda \geq \kappa$
  - $\kappa^\lambda \leq \kappa^\kappa = 2^\kappa$

  So $\kappa \leq \kappa^\lambda \leq 2^\kappa$
- IS THAT IT ?

THEOREM 3.11 IF κ IS INFINITE THEN $\kappa < \kappa^{cf\kappa}$

PROOF: LET $\xi \mapsto \alpha_\xi$ BE INCREASING AND COFINAL FROM $cf(\kappa)$ TO κ.

LET $F \subseteq \kappa^{cf\kappa}$ HAVE CARDINALITY κ, SAY $F = \{f_\alpha : \alpha < \kappa\}$

DEFINE $f : \mathrm{cf}\,\kappa \to \kappa$ BY

$$f(\xi) = \min \kappa \setminus \{ f_\alpha(\xi) : \alpha < \alpha_\xi \}.$$

$(\, |\{ f_\alpha(\xi) : \alpha < \alpha_\xi \}| \leq |\alpha_\xi| < \kappa \,)$

THEN $f \notin F$. $\square$

FIRST PIECE OF EXTRA INFORMATION

- IF $\mathrm{cf}\,\kappa \leq \lambda$ (AND $2^\lambda < \kappa$)
  THEN $\kappa < 2^\lambda$.

USEFUL NOTION AND NOTATION

IF $|A| \geq \lambda$ THEN

$$[A]^\lambda = \{ X \subseteq A : |X| = \lambda \}.$$

$$[A]^{\leq\lambda} = \{ X \subseteq A : |X| \leq \lambda \}.$$

$$[A]^{<\lambda} = \{ X \subseteq A : |X| < \lambda \}$$

LEMMA 5.7 IF $\kappa \geq \lambda$ THEN $|[\kappa]^\lambda| = \kappa^\lambda$

PROOF. $\kappa^\lambda \subseteq [\kappa \times \lambda]^\lambda$ SO $\kappa^\lambda \leq |[\kappa]^\lambda|$

- BLATANT CHOICE: THERE IS AN INJECTION $X \longmapsto f_X$ FROM $[\kappa]^\lambda$ TO κ^λ
  CHOOSE f_X WITH $X = \mathrm{ran}\, f_X$.
- CANTOR-BERNSTEIN THEOREM FINISHES THE PROOF

IF λ IS A LIMIT CARDINAL THEN

$$\kappa^{<\lambda} = \sup\{ \kappa^\mu : \mu < \lambda \} \; \mu \text{ IS A CARDINAL} \}$$

WE SHALL SEE MOMENTARILY

$$\kappa^{<\lambda} = |[\kappa]^{<\lambda}|$$

WE NEED SUMS AND PRODUCTS OF ARBITRARY FAMILIES OF CARDINALS.

LET $\{\kappa_i : i \in I\}$ BE GIVEN

$$\sum_{i \in I} \kappa_i = \left| \bigcup_{i \in I} \{i\} \times \kappa_i \right|$$

$$\underbrace{\prod_{i \in I} \kappa_i}_{\text{CARDINAL}} = \Big| \underbrace{\prod_{i \in I} \kappa_i}_{\text{SET OF FUNCTIONS}} \Big|$$

NOTE: THE INDEXING IS IMPORTANT

WE CAN HAVE $\kappa_i = \kappa_j$ FOR ALL i AND j

AND THEN WE GET $\kappa \cdot |I|$ AND $\kappa^{|I|}$

LEMMA 5.8 IF λ IS INFINITE AND $\kappa_i > 0$ FOR $i \in \lambda$

THEN $\sum_{i \in \lambda} \kappa_i = \lambda \cdot \sup_{i \in \lambda} \kappa_i$

PROOF

- $\bigcup_{i \in \lambda} \{i\} \times \kappa_i \subseteq \lambda \times \mu$ WHERE $\mu = \sup_{i \in \lambda} \kappa_i$

  THIS GIVES "$\leq$".
- $\sum_{i \in \lambda} \kappa_i \geq \sum_{i \in \lambda} 1 = \lambda$
- $\sum_{i \in \lambda} \kappa_i \geq \kappa_j$ FOR ALL j

  SO $\sum_{i \in \lambda} \kappa_i \geq \mu$
- HENCE $\sum_{i \in \lambda} \kappa_i \geq \max\{\lambda, \mu\} = \lambda \cdot \mu$.

SO IF $\lambda \leq \sup_{i \in \lambda} \kappa_i$ THEN

$$\sum_{i \in \lambda} \kappa_i = \sup_{i \in \lambda} \kappa_i$$

AND κ IS SINGULAR IFF $\kappa = \sum_{i \in \lambda} \kappa_i$ WITH $\lambda < \kappa$ AND $\kappa_i < \kappa$ FOR ALL i.

NOW SHOW $\left|[\kappa]^{<\lambda}\right| = \kappa^{<\lambda}$.

PRODUCTS ARE COMMUTATIVE AND ASSOCIATIVE

IF $I = \bigcup_{j\in J} A_j$ (A DISJOINT UNION)

THEN

$$\prod_{i\in I} \kappa_i = \prod_{j\in J} \left(\prod_{i\in A_j} \kappa_i\right)$$

- IF $\kappa_i \geq 2$ FOR ALL $i \in I$ THEN $\sum_{i\in I} \kappa_i \leq \prod_{i\in I} \kappa_i$ ($1+1 \geq 1\cdot 1$)

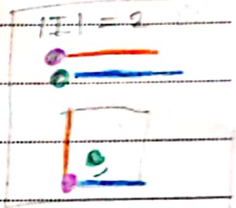
DEFINE $f : \bigcup_{i\in I} \{i\} \times \kappa_i \to \prod_{i\in I} \kappa_i$ ($|I| \geq 3$)

BY

$$f(i,0)(j) = \begin{cases} 0 & j = i \\ 1 & j \neq i \end{cases}$$

$$f(i,\alpha)(j) = \begin{cases} \alpha & j = i \\ 0 & j \neq i \end{cases}$$

$|I| = 2$



THEN f IS INJECTIVE. [CHECK THIS]

LEMMA 5.9 IF $\langle \kappa_i : i \in \lambda \rangle$ IS NON DECREASING ($i < j \to \kappa_i \leq \kappa_j$) AND λ IS INFINITE ($\kappa_i \neq 0$)

THEN $\prod_{i\in\lambda} \kappa_i = (\sup_{i\in\lambda} \kappa_i)^\lambda$.

PROOF [HOMEWORK]

- WE KNOW $\lambda = \lambda \cdot \lambda$ SO VIA A BIJECTION WRITE $\lambda = \bigcup_{j\in\lambda} A_j$ WITH $|A_j| = \lambda$ FOR ALL j AND $A_j \cap A_k = \emptyset$ IF $j \neq k$.

  THEN $\prod_{i\in A_j} \kappa_i \geq \sup_{i\in A_j} \kappa_i = \sup_{i\in\lambda} \kappa_i$

  AND SO

$$\prod_{i\in\lambda} \kappa_i = \prod_{j\in\lambda} \left(\prod_{i\in A_j} \kappa_i\right) \geq (\sup_{i\in\lambda} \kappa_i)^\lambda$$

- CLEARLY

$$\prod_{i\in\lambda} \kappa_i \leq \prod_{i\in\lambda} \kappa = \kappa^\lambda$$

WE DO NOT GET $(\sup_{i\in\lambda} \kappa_i)^\lambda$ IN GENERAL

$\lambda = \aleph_0$ $\quad \kappa_0 = \aleph_0$ $\quad \kappa_n = 1$ $\quad n \geq 1$

$\prod_{i\in\lambda} \kappa_i = \aleph_0$ $\quad (\sup_{i\in\lambda} \kappa_i)^\lambda = 2^{\aleph_0}$

THEOREM [KÖNIG]

IF $\kappa_i < \lambda_i$ FOR ALL i

THEN $\sum_{i\in I}\kappa_i < \prod_{i\in I}\lambda_i$

PROOF

WLOG $\kappa_i \geq 1$ FOR ALL i

[IF $J = \{i : \kappa_i \geq 1\}$ THEN

$$\sum_{i\in I}\kappa_i = \sum_{i\in J}\kappa_i < \prod_{i\in J}\lambda_i \leq \prod_{i\in I}\lambda_i \;]$$

So $\lambda_i \geq 2$ FOR ALL i AND WE GET

$$\sum_{i\in I}\kappa_i \leq \sum_{i\in I}\lambda_i \leq \prod_{i\in I}\lambda_i$$

Now LET

$$F: \bigcup_{i\in I}\{i\}\times\kappa_i \longrightarrow \prod_{i\in I}\lambda_i$$

BE ANY MAP. WE SHOW IT IS NOT SURJECTIVE.

DEFINE $f \in \prod_{i\in I}\lambda_i$ AS FOLLOWS

$$f(i) = \min \lambda_i \setminus \{F(i,\alpha)(i) : \alpha \in \kappa_i\}$$

L PROJECTION ONTO iTH COORD]

THEN $f \neq F(i,\alpha)$ FOR ALL (i,α). □

CONSEQUENCES

1. $\kappa_i = 1$, $\lambda_i = 2$ GIVES US $|I| < 2^{|I|}$
2. $cf(2^\kappa) > \kappa$: (κ INFINITE)

   TAKE $\kappa_i < 2^\kappa$ FOR $i \in \kappa$

   LET $\lambda_i = 2^\kappa$ FOR $i \in \kappa$

   THEN $\sum_{i\in\kappa}\kappa_i < \prod_{i\in\kappa}\lambda_i = (2^\kappa)^\kappa = 2^\kappa$.
3. $cf(\aleph_\alpha^{\aleph_\beta}) > \aleph_\beta$ SAME PROOF
4. $\kappa^{cf\kappa} > \kappa$ SAY $\kappa = \sum_{i\in cf\kappa}\kappa_i$

   WITH $\kappa_i < \kappa$ FOR ALL i

   $$\kappa = \sum_{i\in cf\kappa}\kappa_i < \prod_{i\in cf\kappa}\kappa = \kappa^{cf\kappa}$$

THIS IS WHAT WE CAN SAY BY RELATIVELY ELEMENTARY MEANS.

THE CONTINUUM FUNCTION

WE HAVE $2^{\aleph_\alpha} > \aleph_\alpha$ FOR ALL α

SO $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$

GCH THAT'S IT : $(\forall\alpha)(2^{\aleph_\alpha} = \aleph_{\alpha+1})$

WE KNOW THIS IS CONSISTENT WITH ZFC

IT MAKES LIFE EASY:

THEOREM 5.15 ASSUME GCH

FOR INFINITE κ AND λ WE HAVE

- IF $\kappa \leq \lambda^+$ THEN $\kappa^\lambda = \lambda^+$

  WE ALREADY KNOW THIS $\lambda^+ = 2^\lambda$
- IF $cf\,\kappa \leq \lambda < \kappa$ THEN $\kappa^\lambda = \kappa^+$

  WE ALREADY KNOW $\kappa < \kappa^\lambda \leq 2^\kappa = \kappa^+$
- IF $\lambda < cf\,\kappa$ THEN $\kappa^\lambda = \kappa$.

  FOR $\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda$

  AND $|\alpha^\lambda| \leq 2^{|\alpha|\cdot\lambda} \leq (|\alpha|\cdot\lambda)^+ \leq \kappa$.

  SO $\kappa^\lambda \leq \kappa\cdot\kappa = \kappa$. $\square$

"BUT IN GENERAL?

CANTOR WANTED $2^{\aleph_0} = \aleph_1$

WHAT WE KNOW FOR CERTAIN IS

$2^{\aleph_0} \neq \aleph_\omega,\ \aleph_{\omega+\omega},\ \aleph_{\omega+\omega+\omega}, \ldots$

AND THIS IS IT!

IF $cf\,\aleph_\alpha \neq \aleph_0$ THEN $2^{\aleph_0} = \aleph_\alpha$

IS CONSISTENT WITH ZFC.

SOLOVAY "$2^{\aleph_0}$ CAN BE ANYTHING IT OUGHT TO BE"

ALSO $\aleph_{\omega_1}$

! IF $F : \text{CARD} \to \text{CARD}$
SATISFIES
- $\kappa < \lambda \to F(\kappa) \le F(\lambda)$
- $\mathrm{cf}\, F(\kappa) > \kappa$

THEN THERE IS A MODEL OF SET THEORY WHERE $2^\kappa = F(\kappa)$ FOR ALL REGULAR κ.

WHAT CAN WE SAY?

NOTATION:
- $\beth_0 = \aleph_0$ BETH
- $\beth_{\alpha+1} = 2^{\beth_\alpha}$
- $\beth_\alpha = \sup_{\beta<\alpha} \beth_\beta$ (α LIMIT).

SO GCH IS $\aleph_\alpha = \beth_\alpha$ FOR ALL α.

THEOREM 5.16

(i) $\kappa < \lambda$ IMPLIES $2^\kappa \le 2^\lambda$ $\quad 2^{\aleph_0} = 2^{\aleph_1}$ POSSIBLE

(ii) $\mathrm{cf}\, 2^\kappa > \kappa$

(iii) κ A LIMIT: $2^\kappa = (2^{<\kappa})^{\mathrm{cf}\,\kappa}$.

PROOF: ONLY (iii) NEEDS WORK

SAY $\kappa = \sum_{i<\mathrm{cf}\,\kappa} \kappa_i$ WITH $\kappa_i < \kappa$ FOR ALL i

$$2^\kappa = 2^{\sum_i \kappa_i} = \prod_{i<\mathrm{cf}\,\kappa} 2^{\kappa_i} \le \prod_{i<\mathrm{cf}\,\kappa} 2^{<\kappa}$$
$$= (2^{<\kappa})^{\mathrm{cf}\,\kappa}$$
$$\le (2^\kappa)^{\mathrm{cf}\,\kappa} = 2^\kappa.$$

SO IF κ IS SINGULAR AND 2^μ IS CONSTANT ON AN INTERVAL (λ, κ) THEN 2^κ HAS THAT CONSTANT VALUE

SAY WE HAVE $\mu < \kappa$ AND λ SUCH THAT $\mu > \mathrm{cf}\,\kappa$ AND $2^\nu = \lambda$ FOR $\nu \in [\mu, \kappa)$.

SO $2^{<\kappa} = 2^\mu = \lambda$

AND $2^\kappa = (2^\mu)^{\mathrm{cf}\,\kappa} = 2^\mu = \lambda$.

MORE NOTATION: $\gimel(\kappa) = \kappa^{\mathrm{cf}\,\kappa}$ GIMEL

THIS FUNCTION GOVERNS MOST OF THE BEHAVIOUR OF THE CONTINUUM FUNCTION AND EXPONENTIATION.

ASSUME κ IS A LIMIT CARDINAL

AND $\mu \mapsto 2^{\mu}$ IS NOT CONSTANT

ON ANY INTERVAL $[\lambda, \kappa)$ WITH $\lambda < \kappa$.

THEN $2^{<\kappa} = \sup_{\nu < \kappa} 2^{\nu}$

IT FOLLOWS THAT

$\mathrm{CF}\, 2^{<\kappa} = \mathrm{CF}\, \kappa$

AND $2^{\kappa} = (2^{<\kappa})^{\mathrm{CF}\,\kappa} = \underline{\gimel(2^{<\kappa})}$

ALSO IF κ IS REGULAR THEN $\mathrm{CF}\,\kappa = \kappa$

AND SO $2^{\kappa} = \kappa^{\kappa} = \gimel(\kappa)$.

SUMMARY:

- κ SUCCESSOR: $2^{\kappa} = \gimel(\kappa)$
- κ LIMIT + CONTINUUM FUNCTION EVENTUALLY CONSTANT BELOW κ

  $2^{\kappa} \geq 2^{<\kappa}$; $2^{\kappa} = \kappa^{\kappa} \geq \gimel(\kappa)$

  $2^{\kappa} = 2^{<\kappa} \cdot \gimel(\kappa)$. SINCE $2^{\kappa} = 2^{<\kappa}$; REG $2^{\kappa} = \gimel(\kappa)$.
- κ LIMIT + CONTINUUM FUNCTION NOT EVENTUALLY CONSTANT BELOW κ

  $2^{\kappa} = \gimel(2^{<\kappa})$

EXPONENTIATION

NOTE: IF κ IS REGULAR AND $\lambda < \kappa$

THEN AS SETS $\kappa^{\lambda} = \bigcup_{\alpha < \kappa} \alpha^{\lambda}$

SO AS CARDINALS $\kappa^{\lambda} = \sum_{\alpha < \kappa} |\alpha|^{\lambda}$ <u>VIA SUP</u>

FIRST FORMULA (κ SUCCESSOR)

HAUSDORFF

$$\aleph_{\alpha+1}^{\aleph_{\beta}} = \aleph_{\alpha}^{\aleph_{\beta}} \cdot \aleph_{\alpha+1}$$

IF κ IS A LIMIT CARDINAL AND $\lambda \geq cf\kappa$ THEN

$$\kappa^\lambda = (\sup_{\alpha<\kappa} \alpha^\lambda)^{cf\kappa}$$

SAY $\kappa = \sum_{i<cf\kappa} \kappa_i$ WITH $\kappa_i < \kappa$ FOR ALL i

SO

$$\begin{aligned}
\kappa^\lambda = (\textstyle\sum_{i<cf\kappa} \kappa_i)^\lambda &\leq (\textstyle\prod_{i<cf\kappa} \kappa_i)^\lambda \\
&= \textstyle\prod_{i<cf\kappa} \kappa_i^\lambda \\
&\leq \textstyle\prod_{i<cf\kappa} (\sup_{\alpha<\kappa} \alpha^\lambda) \\
&= (\sup_{\alpha<\kappa} \alpha^\lambda)^{cf\kappa} \\
&\leq (\kappa^\lambda)^{cf\kappa} \\
&= \kappa^\lambda
\end{aligned}$$

SUMMARY: [THEOREM 5.20]

LET λ BE INFINITE THEN κ^λ IS COMPUTED BY INDUCTION ON λ AS FOLLOWS

(i) $\kappa \leq 2^\lambda$ IMPLIES $\kappa^\lambda = 2^\lambda$

(ii) MORE GENERALLY IF THERE IS A $\mu < \kappa$ SUCH THAT $\kappa \leq \mu^\lambda$ THEN $\kappa^\lambda = \mu^\lambda$

(iii) IF $\mu^\lambda < \kappa$ FOR ALL $\mu < \kappa$ (INCLUDING $\mu = \lambda$) THEN: $cf\kappa > \lambda$ IMPLIES $\kappa^\lambda = \kappa$

$cf\kappa \leq \lambda$ IMPLIES $\kappa^\lambda = \kappa^{cf\kappa}$

PROOF (i) KNOWN FROM THE BEGINNING

(ii) SAME PROOF $\mu^\lambda \leq \kappa^\lambda \leq (\mu^\lambda)^\lambda$.

(iii) - $\kappa = \mu^+$: $\kappa^\lambda = \mu^\lambda \cdot \kappa = \kappa$

- κ LIMIT ; SO $\kappa = \sup_{\alpha<\kappa} \alpha^\lambda$

$cf\kappa > \lambda$: SETS $\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda$

SO $\kappa^\lambda = \kappa$

$cf\kappa \leq \lambda$: $\kappa^\lambda = \kappa^{cf\kappa}$

Corollary

The value of κ^λ can be

- 2^λ or
- κ or
- $\mathfrak{I}(\mu)$ for some μ with $cf\mu \leq \lambda < \mu$

if $\kappa^\lambda > 2^\lambda, \kappa$

take μ minimal with $\mu^\lambda = \kappa^\lambda$

then $\nu^\lambda < \mu$ if $\nu < \mu$

- we have $\mu \leq \kappa$ and $\mu^\lambda > \kappa$

so we have $cf\mu \leq \lambda$

- we have $\mu^\lambda > 2^\lambda$ so $\lambda < \mu$.

and we get $\mu^\lambda = \mu^{cf\mu}$.

Most of the research goes into $\kappa^{cf\kappa}$

Strong limit $\lambda < \kappa$ implies $2^\lambda < \kappa$

and so $\mu^\lambda < \kappa$ if $\mu, \lambda < \kappa$

and $2^\kappa = \kappa^{cf\kappa}$

SCH: for every singular κ

<u>if</u> $2^{cf\kappa} < \kappa$ <u>then</u> $\kappa^{cf\kappa} = \kappa^+$

- follows from GCH
- makes life relatively easy

See the Group Interaction