Set Theory

SEVENTH LECTURE

25 October 2021

FINAL LECTURE FOR RUDIMENTS OF AXIOMATIC SET THEORY. THE EXAMINABLE MATERIAL IS CONTAINED IN LECTURES I TO VII.

IF YOU PLAN TO SIT THE RUDIMENTS EXAM ON

MONDAY 20 DECEMBER 2021

17-19

MAKE SURE THAT YOU ARE REGISTERED FOR RUDIMENTS AND LET US KNOW BY EARLY DECEMBER !!

REMINDER I

Zermelo-Fraenkel w/ Choice

Working in $\boxed{ZFC}$, we defined the size of a set

$\longrightarrow$ $|X| :=$ the unique cardinal κ s.t. $X \sim \kappa$

[there is a bijection between X and κ]

Remark on sizes of sets without AC

Can we find other canonical representatives not relying on the Wellordering Theorem.

Frege's idea Why not take the equivalence class $[X]_\sim$?

Disadvantage. These equivalence classes are in general not sets.

There is a way to identify sets that represent that class without AC: Scott's Trick.

Work in ZF. If $[X]_\sim \neq \emptyset$, then there is some α s.t. $[X]_\sim \cap V_{\alpha+1} \neq \emptyset$.

Let α_0 be minimal with this property.

Then
$$|X| := [X]_\sim \cap V_{\alpha_0+1}.$$

If $X \sim Y$, then $[X]_\sim = [Y]_\sim$, and so $|X| = |Y|$.

So these are canonical representatives.

Let's call all of those objects cardinalities.

Z is a cardinality if there is X s.t. $Z = |X|$.

If Z, Z' are cardinalities then we define

$$Z \leq Z' \;:\Longleftrightarrow$$

for all $X \in Z$ and $Y \in Z'$, there is an injection from X to Y.

Observe. $\leq$ behaves like a partial order:

- REFLEXIVE
- TRANSITIVE
- ANTISYMMETRIC

But we do not know whether it is a total order.

As mentioned last time without proof, this relation is total iff $\forall X, Y$ there is an inj. from $X \longrightarrow Y$ or from $Y \longrightarrow X$ iff AC.

Reminder II We had identified regular and singular cardinals:

a cardinal κ is regular if there is no subset $C \subseteq \kappa$ s.t.

$$|C| < \kappa \text{ and } \bigcup C = \kappa.$$

It is singular otherwise.

We proved in ZFC: AC is needed here!

$$\boxed{\text{Every successor cardinal } \aleph_{\alpha+1} \text{ is regular.}}$$

And we observed that many limit cardinals are not:

$$\aleph_\omega = \bigcup \underbrace{\{\aleph_n ; n \in \mathbb{N}\}}_{\text{size of this set is } \aleph_0 < \aleph_\omega}$$

$$\aleph_{\omega_1} = \bigcup \underbrace{\{\aleph_\alpha ; \alpha < \omega_1\}}_{\text{size of this set is } \aleph_1 < \aleph_{\omega_1}}$$

$$\aleph_{\omega_2} = \bigcup \underbrace{\{\aleph_\alpha ; \alpha < \omega_2\}}_{\text{size of this set is } \aleph_2 < \aleph_{\omega_2}}$$

Observation

If κ is an uncountable limit cardinal and κ is regular, then

$$\kappa = \aleph_\kappa.$$

[Suppose not: $\kappa \neq \aleph_\kappa \Longrightarrow \kappa < \aleph_\kappa$.

$\kappa = \aleph_\lambda < \aleph_\kappa$ $\lambda < \kappa$ & λ is a limit ordinal.

$$\aleph_\lambda = \bigcup \underbrace{\{\aleph_\alpha ; \alpha < \lambda\}}_{\text{has size } \lambda < \kappa}$$

So $\aleph_\lambda = \kappa$ is singular. Contradiction!]

Q. Can there be any such κ s.t. $\kappa = \aleph_\kappa$?

A. **Yes!** We proved that normal ordinal operations have arbitrarily large fixed pts.

So, there are cardinals $\kappa = \aleph_\kappa$.

What is the smallest such fixed pt?

[These are called aleph fixed pt.]

$$\alpha_0 := \omega$$

$$\alpha_{n+1} := \aleph_{\alpha_n}$$

$$\alpha := \boxed{\bigcup \{\alpha_n ; n \in \mathbb{N}\}}$$

$$\underbrace{\aleph_0}_{\alpha_0}, \underbrace{\aleph_\omega}_{\alpha_1}, \underbrace{\aleph_{\aleph_\omega}}_{\alpha_2}, \underbrace{\aleph_{\aleph_{\aleph_\omega}}}_{\alpha_3}, \ldots$$

Proof shows that α is an aleph fixed pt, so $\aleph_\alpha = \alpha$.

Clearly α is not countable.

But by definition, α is a union of a countable set; so it's singular.

[It is singular in a very strong way: it is a countable union of smaller things.]

(Measuring the degree of singularity — we'll do this in a moment

SUMMARY

	LIMIT CARDS	SUCCESSOR CARDS
REGULAR	?	✓
SINGULAR	✓	✗

Q. Are there any regular limit cardinals?

If they do, they must be gigantic!

__Def.__ A cardinal κ is called __weakly inaccessible__ if it is an uncountable regular limit cardinal.

It turns out that ZFC $\nvdash$ there are weakly inaccessible cardinals.

[No proof in this lecture, but we'll prove that ZFC $\nvdash$ there are strongly inacc. cardinals.]

If λ is a limit ordinal and $C \subseteq \lambda$, we say that C is cofinal or unbounded iff $\bigcup C = \lambda$.
[Eq. to $\forall \alpha \in \lambda \exists \gamma \in C (\alpha < \gamma)$.]

Def. $cf(\lambda) := \min \{ |C| ; C \text{ is cofinal in } \lambda \}$

This is a class of ordinals, non-empty since λ is unbounded in λ and thus a minimal elt. exists.

COFINALITY OF λ

$\Longrightarrow cf(\lambda) \leq |\lambda|$.

If $f : \alpha \longrightarrow \lambda$, we call f cofinal if the range of f is a cofinal subset of λ.

Group Interaction #6

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2021/22: 1st Semester
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(2) If λ is a limit ordinal and κ is a cardinal, prove that the following statements are equivalent:

(a) There is a cofinal increasing function $f : \kappa \to \lambda$ (i.e., if $\alpha < \beta$, then $f(\alpha) < f(\beta)$).

(b) There is a cofinal subset of λ of size κ.

Use this to prove: for each λ, $cf(\lambda)$ is a regular cardinal.

Using this notation, we proved that $cf(\kappa) = \aleph_0$ if κ is the smallest aleph fixed pt.

[GI #6 constructs aleph fixed pts of larger cofinality:

e.g. $\kappa = \aleph_\kappa$ with $cf(\kappa) = \aleph_1$

$\aleph_2$

$\aleph_3$

$\vdots$]

In lecture $\overline{\text{VIII}}$, we'll cover more on the cofinality function.

Cardinal Arithmetic

ARITHMETIC Lecture III, page 9

How do we add natural numbers?
Two very different definitions that end up being equivalent:

SYNTHETIC DEFINITION
CARDINAL DEFINITION

Define for sets X and Y a disjoint union:

$$X \uplus Y := \{0\} \times X \cup \{1\} \times Y$$

Remark There there nothing canonical about this definition and it implies in general that $X \not\subseteq X \uplus Y$.

Define a function

$$\oplus : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$

by $n \oplus m = k$ if and only if k is the unique number in bij. with $n \uplus m$.

Note: We would need to prove that such a number exists and is unique. (cf. HW (10)).

INDUCTIVE DEF.

$$n+0 := n$$
$$n+(m+1) := (n+m)+1$$

$\downarrow$

ORDINAL ARITHMETIC

? Synthetic ordinal addition ??

α, β $\qquad \alpha \uplus \beta$

If α, β are infinite, then there is no unique ordinal γ s.t. $\gamma \sim \alpha \uplus \beta$.

Example

If $\alpha = \beta = \omega$, then

$\omega \uplus \omega$ is countable,

so every α s.t. $\boxed{\omega \leq \alpha < \omega_1}$ is in bijection with $\omega \uplus \omega$.

But There is a unique cardinal κ s.t. $\kappa \sim \alpha \uplus \beta$

So we define for cardinals κ, λ:

$\kappa + \lambda := \mu$ if μ is the unique cardinal s.t. $|\kappa \uplus \lambda| = \mu$.

$\kappa + \lambda := |\kappa \uplus \lambda|$

$\kappa \cdot \lambda := \mu$ if μ is the unique cardinal s.t. $|\kappa \times \lambda| = \mu$.

$\kappa \cdot \lambda := |\kappa \times \lambda|$.

WARNING We're defining operations $+$ and $\cdot$ on cardinals, but they already have operations $+$ and $\cdot$: ordinal arithmetic operations.

Just for specificity write

$\boxplus$ for cardinal $+$

and $\diamondplus$ for ordinal $+$

$\boxdot$ for cardinal $\cdot$

$\diamonddot$ for ordinal $\cdot$

$$\omega \boxplus \omega = \omega \neq \omega \diamondplus \omega = \omega \diamonddot 2$$

$$\omega \diamonddot 2 \neq \omega \boxdot 2 = \omega.$$

So, cardinal $+, \cdot$ and ordinal $+, \cdot$ are the same symbols with different meaning.

Whenever you see $+, \cdot$, you need to ask yourself: which one is it?

One way to disambiguate:

use κ, λ, μ for cardinals
$\aleph_\alpha$

use $\alpha, \beta, \gamma, \delta$ for ordinals
ω_α

and mean cardinal operations if you use the former and ordinal operations if you use the latter.

From GI#5, we have already learned that cardinal $+$ and $\cdot$ are not very interesting operations (on infinite cardinals):

$\omega \leq \kappa$ or $\omega \leq \lambda$

$$\Longrightarrow$$

$$\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda).$$

What about exponentiation?

HW #5 (17) gave us a synthetic notion of exponentiation:

Homework Sheet #5

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(17) Suppose α and β are ordinals and define $\underline{F(\beta,\alpha)} := \{f : \beta \to \alpha;$ for all but finitely many $\gamma \in \beta$, $f(\gamma) = 0\}$ Define an order $\prec$ on $F(\beta,\alpha)$ by

$$f \prec g :\iff f(\mu) < g(\mu) \text{ where } \mu := \max\{\gamma \in \beta;\ f(\gamma) \neq g(\gamma)\}.$$

Show that $\boxed{(F(\beta,\alpha),\prec) \cong (\alpha^\beta, \in)}$.

One idea could be to define

$$\kappa^\lambda \cancel{:=} |F(\lambda,\kappa)|$$

This is not done!
Instead, we are going to define κ^λ in a way that is entirely different!

Think about $F(\lambda,\kappa)$ with $\lambda,\kappa \in \mathbb{N}$:

$$F(n,m) = \{f : n \to m\ ;\ f \text{ is a function}\}$$

What is $|F(n,m)| = \underline{m^n}$.

Standard notion for functions from X to Y:

$$Y^X := \{f;\ f : X \to Y\}$$

OFFICIAL DEFINITION OF CARDINAL EXPONENTIATION:

$$\kappa^\lambda := |\{f;\ f : \lambda \to \kappa\}|.$$

Cardinal exponentiation gives us something that is fundamentally different from ordinal exponentiation:

Simplest possible case:

$$\kappa = 2$$
$$\lambda = \omega.$$

ORDINAL EXPONENTIATION:

$$2^\omega = \bigcup\{2^\alpha ; \alpha < \omega\}$$
$$= \bigcup\{2^n ; n \in \mathbb{N}\}$$
$$= \omega.$$

CARDINAL EXPONENTIATION:

$2^{\aleph_0}$ is uncountable.

This follows from Cantor's Theorem.

Theorem (Cantor).

If X is any set, then there is no surjection from X onto $\{f ; f : X \longrightarrow 2\} = 2^X$.

Remark So no surj. from $\aleph_0$ to $2^{\aleph_0}$.

Proof. Suppose $F : X \longrightarrow 2^X$.

We construct $f \in 2^X$ s.t. $f \notin \mathrm{ran}(F)$.

$$f(x) := 1 - F(x)(x).$$

Suppose $f \in \mathrm{ran}(F)$, i.e., there is $x_0 \in X$ s.t. $f = F(x_0)$.

$$\underline{f(x_0)} = 1 - F(x_0)(x_0)$$
$$= \underline{1 - f(x_0)}$$

Contradiction! q.e.d.

So if κ, λ cardinals, κ^λ can mean

1. ordinal exponentiation
2. the set of fns from λ to κ
3. the cardinal κ^λ, i.e., the cardinality of 2.

Homework Sheet #7

(24) Let κ, λ, μ, and ν be non-zero cardinals. Show the following rules of *cardinal arithmetic* by providing explicit bijections or injections between the corresponding sets:

(a) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$,

(b) $\kappa^\lambda \cdot \kappa^\mu = \kappa^{\lambda+\mu}$, and

(c) $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.

(d) If $\kappa \leq \lambda$ and $\mu \leq \nu$, then $\kappa^\mu \leq \lambda^\nu$.

and κ infinite

Lemma If $2 \leq \lambda \leq \kappa$, then $\lambda^\kappa = \kappa^\kappa$.

Proof. We know $\kappa \leq 2^\kappa$.

$$\boxed{\kappa^\kappa} \underset{(d)}{\leq} (2^\kappa)^\kappa \underset{(c)}{=} 2^{\kappa \cdot \kappa} = 2^{\max(\kappa,\kappa)}$$
$$= 2^\kappa$$
$$\underset{(d)}{\leq} \lambda^\kappa$$
$$\underset{(d)}{\leq} \boxed{\kappa^\kappa}$$

So $2^\kappa = \lambda^\kappa = \kappa^\kappa$.

q.e.d.

Def. If κ is a cardinal and $\kappa = \aleph_\alpha$, we write κ^+ for $\aleph_{\alpha+1}$.

Called the cardinal successor of κ.

Lemma If κ is infinite,

$$(\kappa^+)^\kappa = \kappa^\kappa.$$

Proof. By Cantor's Theorem $\kappa^+ \leq 2^\kappa$.

$$\boxed{\kappa^\kappa} \leq (\kappa^+)^\kappa \leq (2^\kappa)^\kappa = 2^{\kappa \cdot \kappa}$$
$$= 2^\kappa \leq \boxed{\kappa^\kappa}.$$

So $(\kappa^+)^\kappa = \kappa^\kappa$. q.e.d.

Corollary

$$\aleph_1^{\aleph_0} = 2^{\aleph_0}.$$

What about $\aleph_2^{\aleph_0}$?

This requires knowing what the relationship between $\aleph_2$ and $2^{\aleph_0}$ is !!

The Continuum Problem

Cantor observed that we have two size-increasing operations:

$$X \to \aleph(X) \quad \text{[Hartogs' Theorem]}$$

$$X \to 2^X \quad \text{[Cantor's Theorem]}$$

Cantor: Do these give the same cardinals?

Cantor conjectured: YES!

CH

CONTINUUM HYPOTHESIS: $2^{\aleph_0} = \aleph_1$.

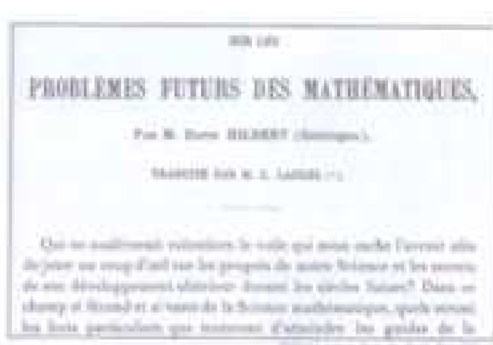
GENERALISED CONTINUUM HYPOTHESIS

GCH

$\forall \kappa \; 2^\kappa = \kappa^+$.

David Hilbert (1862-1943)

Hilbert's Problems



PROBLÈMES FUTURS DES MATHÉMATIQUES,

Hilbert's Problem #1: Is CH provable?

Answer to Hilbert #1:
CH is neither provable nor disprovable in ZFC.

Kurt Gödel

Born	Kurt Friedrich Gödel April 28, 1906 Brünn, Austria-Hungary (now Brno, Czech Republic)
Died	January 14, 1978 (aged 71) Princeton, New Jersey, U.S.

<u>1938</u> $\mathrm{Cons}(ZFC) \Longrightarrow \mathrm{Cons}(ZFC + GCH)$

So $ZFC \nvdash \neg GCH$.

[Same work has $\mathrm{Cons}(ZF) \Longrightarrow \mathrm{Cons}(ZFC)$.]

1962: $\mathrm{Cons}(ZFC) \Longrightarrow \mathrm{Cons}(ZFC + \neg CH)$

(*) So $ZFC \nvdash CH$.

Topic of the final part of the course!

Paul Cohen

American mathematician

Paul Joseph Cohen was an American mathematician. He is best known for his proofs that the continuum hypothesis and the axiom of choice are independent from Zermelo–Fraenkel set theory, for which he was awarded a Fields Medal. Wikipedia

Born: 2 April 1934, Long Branch, New Jersey, United States

Died: 23 March 2007, Stanford, California, United States

Known for: Cohen forcing; Continuum hypothesis

Fields: Mathematics

APPLICATION

THE QUESTION OF INACCESSIBLE CARDINALS

κ is <u>weakly</u> <u>inaccessible</u> if it's a uncountable and regular limit cardinal $[\Leftrightarrow \forall \lambda < \kappa \, (\lambda^+ < \kappa)]$

κ is <u>strongly</u> <u>inaccessible</u> if it's uncountable and regular and $\forall \lambda < \kappa \, (2^\lambda < \kappa)$.

[STRONG LIMIT]

<u>Observation</u> ① GCH implies that every weakly inacc. is strongly inaccessible.

② And ZFC proves the other direction.

Write <u>IC</u> for
$$\exists \kappa \, (\kappa \text{ is strongly inaccessible})$$

We'll show:

if $\mathrm{Cons}(\mathrm{ZFC})$, then $\mathrm{ZFC} \nvdash \mathrm{IC}$.

We'll show that

$$ZFC + IC \vdash \text{there is a model of } ZFC$$

By the usual Gödel Incompleteness Argument that were used for

$$Z \nvdash ZF$$

we get that $ZFC \nvdash IC$ [unless it is inconsistent].

Lemma 1 Assume $ZFC + IC$. Let κ be any strongly inaccessible cardinal, then $V_\kappa \models ZFC$.

Proof. We already know $V_\kappa \models Z$.
AC in V_κ is just the usual rank argument (cf. HW #7).

Critical question: Replacement!

Lemma 2 Suppose κ is strongly inaccessible and $\alpha < \kappa$, then

$$|V_\alpha| < \kappa.$$

Proof. By induction.

① $V_0 = \emptyset$

$$|V_0| = 0 < \kappa.$$

② Suppose $|V_\alpha| < \kappa$. Then

$$|V_{\alpha+1}| = |\mathcal{P}(V_\alpha)| = |2^{V_\alpha}|$$
$$= 2^{|V_\alpha|}$$
$$< \kappa$$

by assumption that κ is a strong limit.

③ Let λ be limit ordinal $\lambda < \kappa$.

$$|V_\lambda| = \left|\bigcup \underbrace{\{V_\alpha ; \alpha < \lambda\}}\right|$$

By IH, for each $\alpha < \lambda$, $|V_\alpha| < \kappa$.

$\{|V_\alpha| ; \alpha < \lambda\} \subseteq \kappa$ has size $\leq \lambda$, so by regularity bounded.

$$V_\lambda = \bigcup \{V_\alpha ; \alpha < \lambda\}$$

Find $\mu < \kappa$ s.t. f.a. $\alpha < \lambda$

$$|V_\alpha| < \mu.$$

$$|V_\lambda| \leq \mu \cdot \lambda = \max(\mu, \lambda) < \kappa.$$

q.e.d.

I am going to prove something much stronger than Replacement in V_κ:

SOR

$$\boxed{\begin{array}{l}\text{If } F : V_\kappa \longrightarrow V_\kappa \text{ and } x \in V_\kappa, \\ \text{then } F[x] \in V_\kappa.\end{array}}$$

This talks about <u>ALL</u> functions whereas Replacement only needs definable functions.

SECOND ORDER REPLACEMENT

Lemma 3 V_κ satisfies SOR.

Proof. Suppose $F : V_\kappa \longrightarrow V_\kappa$ and $x \in V_\kappa$.

Consider $F[x]$. Need to show $F[x] \in V_\kappa$.

By Lemma 2, we know that

$x \in V_\alpha$ for some $\alpha < \kappa$

$\Longrightarrow x \subseteq V_\alpha$ [transitivity of V_α]

$\Longrightarrow |x| \leq |V_\alpha| \underset{L2}{<} \kappa$.

Therefore $|F[x]| \leq |x| < \kappa$.

Consider

$$\begin{cases} C := \{\rho(y)+1 \,;\, y \in \underline{F[x]}\} \subseteq \kappa \\ |C| \leq |F[x]| < \kappa \end{cases}$$

$\longrightarrow$ by regularity of κ: C is bounded

Find bound μ. By definition of C,

$F[x] \subseteq V_\mu$. Thus $F[x] \in V_{\mu+1} \subseteq V_\kappa$.

q.e.d.

Back to the proof of L1:

This yields that

$$\mathrm{ZFC} + \mathrm{IC} \vdash \text{"there is a } \kappa \text{ s.t. } V_\kappa \models \mathrm{ZFC}\text{"}.$$

q.e.d.

__Remark__ The same argument works with

$$\mathrm{WIC} := \exists \kappa (\kappa \text{ is weakly inaccessible})$$

under GCH:

$$\mathrm{ZFC} + \mathrm{GCH} + \mathrm{WIC} \vdash \text{there is a model of } \mathrm{ZFC} + \mathrm{GCH}$$

[Homework #7]

Together: __if ZFC + GCH is consistent,__ *

then $\mathrm{ZFC} + \mathrm{GCH} \nvdash \mathrm{WIC}$.

$$\Longrightarrow \mathrm{ZFC} \nvdash \mathrm{WIC}.$$

[__Remark__. * is what Gödel (1938) proved from Con(ZFC).]

M

$M \models \varphi$ one fixed formula

$\exists x \forall y$ ---

$\Downarrow$

$\exists x \in M \, \forall y \in M$ --

$\Downarrow$

$\boxed{M \models Z}$