SET THEORY Lecture VI

18 October 2021

REMINDER 1

Hartogs's Theorem

$\boxed{\aleph(X)}$: the least ordinal α s.t. α does not inject into X

= the set of all ordinals injecting into X

Lemma 3.4.

(i) *For every α there is a cardinal number greater than α.*
(ii) *If X is a set of cardinals, then $\sup X$ is a cardinal.*

For every α, let α^+ be the least cardinal number greater than α, the *cardinal successor* of α.

Proof. (i) For any set X, let

(3.11) $h(X)$ = the least α such that there is no one-to-one function of α into X.

There is only a set of possible well-orderings of subsets of X. Hence there is only a set of ordinals for which a one-to-one function of α into X exists. Thus $h(X)$ exists.

$\boxed{\omega_1} := \boxed{\aleph_1} := \aleph(\mathbb{N})$

the least uncountable ordinal

= the set of countable ordinals

REMINDER 2

The von Neumann hierarchy

The Cumulative Hierarchy of Sets

We define, by transfinite induction,

$$\boxed{\begin{aligned} &V_0 = \emptyset, \qquad V_{\alpha+1} = P(V_\alpha), \\ &V_\alpha = \bigcup_{\beta<\alpha} V_\beta \quad \text{if } \alpha \text{ is a limit ordinal.}\end{aligned}}$$

The sets V_α have the following properties (by induction):

(i) Each V_α is transitive.
(ii) If $\alpha < \beta$, then $V_\alpha \subset V_\beta$.
(iii) $\alpha \subset V_\alpha$.

$ZF \vdash \forall x \exists \alpha \; x \in V_\alpha$

If $\lambda > \omega$ is a limit ordinal, then $V_\lambda \models Z$

GI#4

Just from $V_{\omega+\omega} \models Z$ and Gödel's Incompleteness Thm, we know that (under reasonable assumptions) $Z \neq ZF$.

<u>Gödel's Incompleteness Thm</u>

If T is "reasonable" and consistent, then $T \nvdash Cons(T)$.

By Gödel's Completeness Thm, this means:

$$T \nvdash \exists M (M \models T).$$

But we proved (in ZF) that $\underbrace{V_{\omega+\omega} \models Z}_{\Phi}$

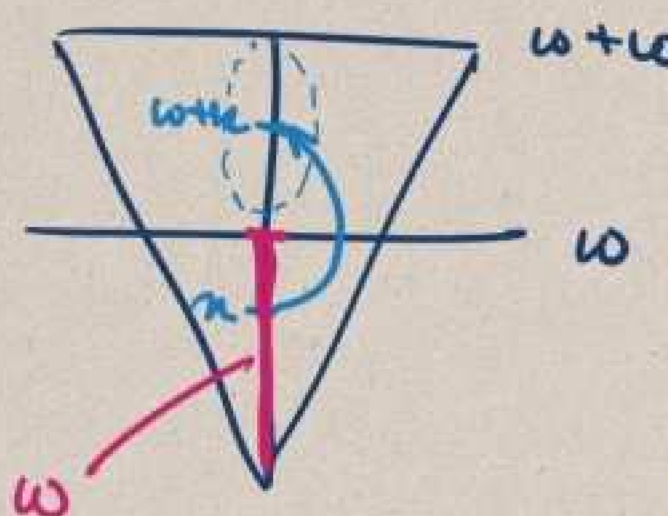
$\Longrightarrow Cons(Z)$

Thus: $ZF \vdash \Phi$

$ZF \vdash Cons(Z)$.

Thus if $ZF = Z$, then Z is inconsistent.

We proved ZH Replacement in a diff. way: we refuted an instance of Repl. in $V_{\omega+\omega}$:



$n \longmapsto \omega+n$

Replacement implies $\{\omega+n \,;\, n\in\mathbb{N}\}$ is a set.

$$\varrho(\{\omega+n; n\in\mathbb{N}\}) = \omega+\omega,$$
$$\text{so } \{\omega+n; n\in\mathbb{N}\} \notin V_{\omega+\omega}.$$

Remark This also refutes the Representation Thm for wellorders in $V_{\omega+\omega}$: if $R \subseteq \mathbb{N}\times\mathbb{N}$ s.t.

$$(\mathbb{N},R) \cong (\omega+\omega, \in)$$

then $(\mathbb{N},R) \in V_{\omega+\omega}$.

But $\omega+\omega \in V_{\omega+\omega+1} \setminus V_{\omega+\omega}$.

The underlying reason here:
we find $x \in V_{\omega+\omega}$ bounded below $\omega+\omega$ s.t. the range of the operation restricted to x is unbounded in rank.

For meta-mathematical reasons (Gödel's incompleteness theorem), we know that it's impossible to have

$$ZF \vdash \exists\alpha(V_\alpha \models ZF).$$

What about ω_1?
Clearly, the previous proof works if there is $f: \mathbb{N} \longrightarrow \alpha$ s.t. $\mathrm{ran}(f)$ is unbounded in α.

<u>Q.</u> Is that the case for ω_1?
[This is (somewhat surprisingly) related to the Axiom of Choice
$\longrightarrow$ later today.]

So, how do we show

$$V_{\omega_1} \not\models ZF.$$

<u>Idea</u> (proof of Hartogs's Thm).

If α is a countable ordinal, by the proof of Hartogs's Thm, there is $R \subseteq \mathbb{N} \times \mathbb{N}$ s.t.

$$A \subseteq \mathbb{N} \quad \underset{\in V_{\omega+\omega}}{(A, R)} \cong (\alpha, \in)$$

$$(A, R) \longmapsto \text{the unique } \alpha \text{ s.t. } (A, R) \cong (\alpha, \in)$$

By Repl., the range of this (functional) formula is a set. But this is the set of all ctble ordinals, so $\omega_1 \notin V_{\omega_1}$.

The Alephs

$\aleph_0 := \omega$

$\aleph_{\alpha+1} := \aleph(\aleph_\alpha)$

$\aleph_\lambda := \bigcup_{\alpha<\lambda} \aleph_\alpha$

λ limit

Using Lemma 3.4, we define the increasing enumeration of all alephs. We usually use $\aleph_\alpha$ when referring to the cardinal number, and ω_α to denote the order-type:

$$\aleph_0 = \omega_0 = \omega, \qquad \aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_\alpha^+, \quad \aleph(\aleph_\alpha)$$

$$\aleph_\alpha = \omega_\alpha = \sup\{\omega_\beta : \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit ordinal.}$$

Sets whose cardinality is $\aleph_0$ are called *countable*; a set is *at most countable* if it is either finite or countable. Infinite sets that are not countable are *uncountable*.

A cardinal $\aleph_{\alpha+1}$ is a *successor cardinal*. A cardinal $\aleph_\alpha$ whose index is a limit ordinal is a *limit cardinal*.

Addition and multiplication of alephs is a trivial matter, due to the following fact:

Theorem 3.5. $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.

An ordinal γ is called an aleph if there is some α s.t. $\gamma = \aleph_\alpha$.

Def. An ordinal γ is called an initial ordinal or cardinal if for $\alpha < \gamma$ there is no bijection between α and γ.

Obs. 1 If γ is the Hartogs aleph of some set X, then it is an initial ordinal.

Obs 2 ω is an initial ordinal.

Obs 3 If λ is a limit ordinal, then $\aleph_\lambda$ is an initial ordinal.

[Suppose not: so there is some $\alpha < \aleph_\lambda$ s.t. α and $\aleph_\lambda$ are in bijection.

Since $\aleph_\lambda = \bigcup_{\gamma<\lambda} \aleph_\gamma$, we find $\gamma < \lambda$ s.t. $\alpha \in \aleph_\gamma \subseteq \aleph_\lambda$

If there is a bij. between $\aleph_\lambda$ and α, this gives an injection from

$$\underset{\aleph(\aleph_\gamma)}{\overset{\aleph_{\gamma+1}}{\shortparallel}} \text{ into } \alpha$$

Contradiction!]

Theorem Let α be an infinite ordinal. Then TFAE:

(i) α is an initial ordinal

(ii) α is an aleph.

Proof. (ii) $\Rightarrow$ (i) is Obs 1., 2., & 3.

(i) $\Rightarrow$ (ii). Suppose not, so let α be initial but not an aleph. Suppose α is least with this property.

α is infinite, $\alpha \neq \omega$, so $\alpha > \omega$.
Consider

$$\{\aleph_\gamma ; \aleph_\gamma \in \alpha\} \text{ and}$$

$$C := \{\gamma ; \aleph_\gamma \in \alpha\}.$$

<u>Case 1.</u> C has no largest element.
Let $\lambda := \bigcup C$. That is a limit ordinal.

$$\aleph_\lambda = \bigcup_{\gamma<\lambda} \aleph_\gamma = \bigcup_{\gamma\in C} \aleph_\gamma = \alpha.$$

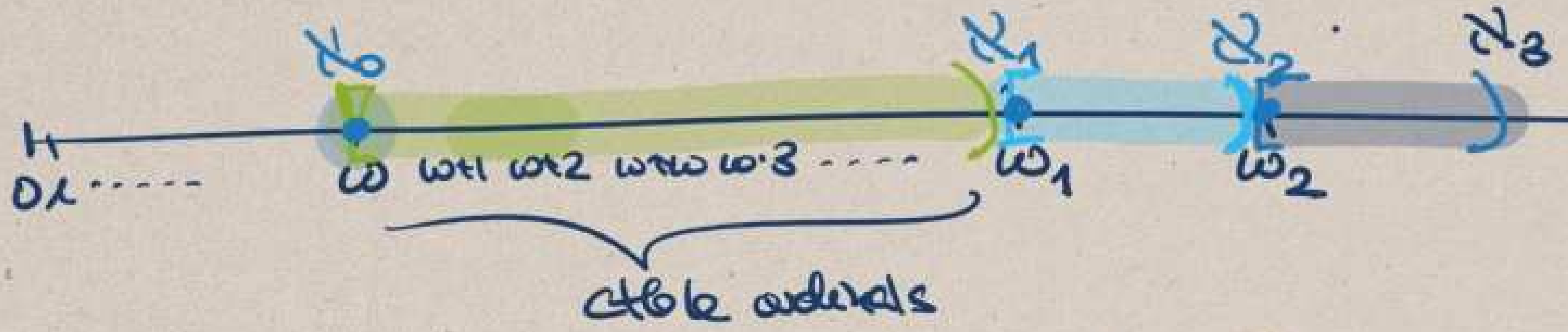
<u>Case 2.</u> C has a largest element, say γ_0: $\aleph_{\gamma_0} \in \alpha$.
Then $\aleph(\aleph_{\gamma_0})$ is the smallest initial ordinal bigger that $\aleph_{\gamma_0}$, so $\underset{\aleph_{\gamma_0+1}}{\underset{\shortparallel}{\aleph(\aleph_{\gamma_0})}} \leq \alpha$.

By maximality, $\alpha = \aleph_{\gamma_0+1}$.

q.e.d.

Successor cardinal vs limit cardinals

$$\aleph_0, \aleph_1, \aleph_2, \aleph_3, \aleph_4$$



For a successor $\aleph_{\alpha+1}$, the set

$$\{\beta \,;\, \aleph_\alpha \leq \beta < \aleph_{\alpha+1}\}$$

consists of ordinals all in bijection with each other. The initial ordinal is the smallest element of this set.

Example $\qquad \aleph_\omega = \bigcup_{n \in \mathbb{N}} \aleph_n$

Note that $\aleph_\omega$ has no immediate cardinal predecessor <u>and</u> it is a countable union of things that are smaller than it.

Def. If X and Y are sets, we say they are

of the same size
equinumerous
equipollent
of equal cardinality

if there is a bijection between X and Y.

[Reminder: this is the notion of size that is used when showing that $\mathbb{R}$ are uncountable. We prove that there is no surjection from $\mathbb{N}$ onto $\mathbb{R}$.]

Jech writes for the above relation

$$|X| = |Y|.$$

This notation is highly suggestive one expects that we find a canonical representative in each eq. class of the eq. rel.

$$X \sim Y \;:\Longleftrightarrow\; \text{there is a bijection between } X \text{ and } Y.$$

Def. X <u>has at most the size</u> of Y
if there is an injection from X to Y.

<u>Jech</u>: $|X| \leq |Y|$.

<u>Small issue</u> You would hope that if

$$|X| \leq |Y| \ \& \ |Y| \leq |X|,$$

then $|X| = |Y|$.

This means: if there is an inj. from X to Y and an inj. from Y to X, then there is a bij. from X to Y.

Cantor-Schröder-Bernstein Theorem

Theorem 3.2 (Cantor-Bernstein). *If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

Proof. If $f_1 : A \to B$ and $f_2 : B \to A$ are one-to-one, then if we let $B' = f_2(B)$ and $A_1 = f_2(f_1(A))$, we have $A_1 \subset B' \subset A$ and $|A_1| = |A|$. Thus we may assume that $A_1 \subset B \subset A$ and that f is a one-to-one function of A onto A_1; we will show that $|A| = |B|$.

We define (by induction) for all $n \in \boldsymbol{N}$:

$$\begin{aligned} A_0 &= A, \qquad & A_{n+1} &= f(A_n), \\ B_0 &= B, \qquad & B_{n+1} &= f(B_n). \end{aligned}$$

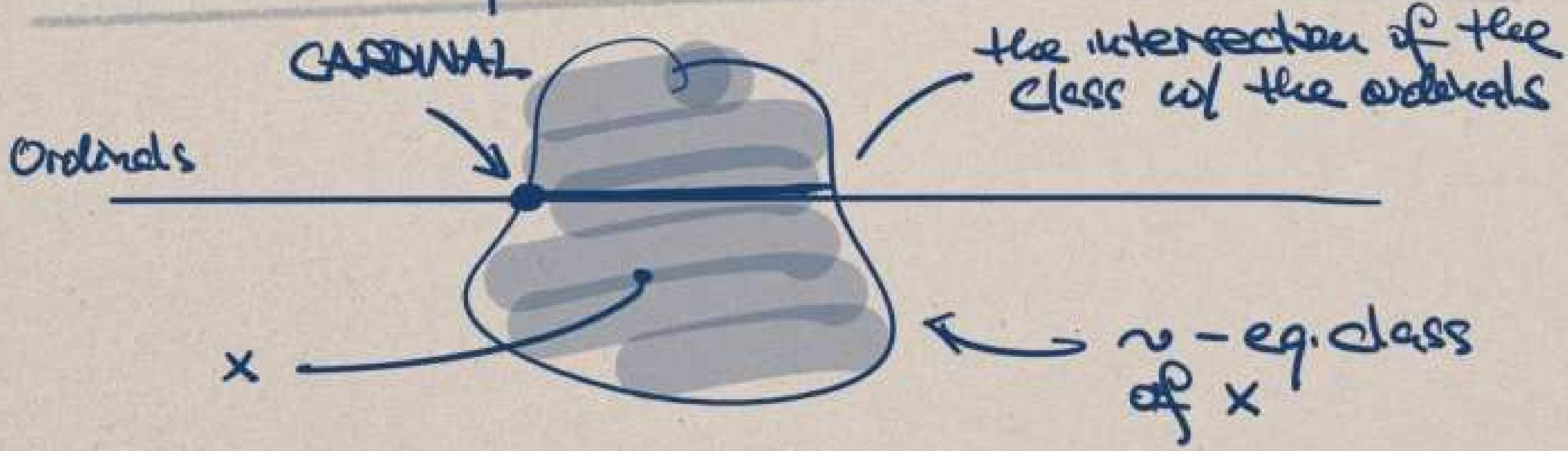
Let g be the function on A defined as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in A_n - B_n \text{ for some } n, \\ x & \text{otherwise.} \end{cases}$$

Then g is a one-to-one mapping of A onto B, as the reader will easily verify. Thus $|A| = |B|$. $\square$

HW #6
with a different proof.

Objects	Eq. rel.	Canonical repr.
Wellorders	$\cong$	ORDINALS
Sets	$\sim$	CARDINALS



Cardinals as canonical representatives in $\sim$-equivalence classes.

Need to show:

① if κ, λ are cardinals and $\kappa \sim \lambda$, then $\kappa = \lambda$.

[Follows from definition of initial ordinal.]

② Every $\sim$-eq. class contains a cardinal.

[② is eq. to "every $\sim$-eq. class contains an ordinal]

Def. A set X is called wellorderable if there is some $R \subseteq X \times X$ s.t. (X, R) is a wellorder.

Proposition Let X be a set. Then TFAE:

(i) X is wellorderable

(ii) there is an ordinal α s.t.

$$\alpha \sim X.$$

Proof. (i) $\Rightarrow$ (ii) Just the representation theorem for wellorder.

(ii) $\Rightarrow$ (i). That's our usual coding trick:

Fix $f : \alpha \longrightarrow X$ bijection.
Define $R_f \subseteq X \times X$ by

$$x R_f y :\Longleftrightarrow f^{-1}(x) \in f^{-1}(y).$$

Then by construction

$$f : (\alpha, \in) \xrightarrow{\cong} (X, R_f).$$

q.e.d.

Therefore, in order to be able to use cardinals as canonical representatives of $\sim$-eq.cl. all I need is

"Every set is wellorderable."

Zermelo 1904 proved what we call

ZERMELO'S WELLORDERING THM

This is not a ZF-theorem but used the Axiom of Choice.

1904

Beweis, daß jede Menge wohlgeordnet werden kann.

(Aus einem an Herrn Hilbert gerichteten Briefe.)

Von

E. Zermelo in Göttingen.

... Der betreffende Beweis ist aus Unterhaltungen entstanden, die ich in der vorigen Woche mit Herrn Erhard Schmidt geführt habe, und ist folgender.

1) Es sei M eine beliebige Menge von der Mächtigkeit $\mathfrak{m}$, deren Elemente mit m bezeichnet werden mögen, M' von der Mächtigkeit $\mathfrak{m}'$ eine ihrer Teilmengen, welche mindestens ein Element m enthalten muß, aber auch alle Elemente von M umfassen darf, und $M - M'$ die zu M' „komplementäre" Teilmenge. Zwei Teilmengen gelten als verschieden, wenn eine von beiden irgend ein Element enthält, das in der anderen nicht vorkommt. Die Menge aller Teilmengen M' werde mit M bezeichnet.

MATHEMATISCHE ANNALEN 59 (1904) pp 514-516

MATHEMATISCHE ANNALEN 65 (1908) pp. 107-128

Neuer Beweis für die Möglichkeit einer Wohlordnung.

Von

E. Zermelo in Göttingen.

Obwohl ich meinen im Jahre 1904 veröffentlichten „Beweis, daß jede Menge wohlgeordnet werden kann"*) gegenüber den verschiedenen im § 2 ausführlich zu besprechenden Einwendungen noch heute vollkommen aufrecht erhalte, dürfte doch der hier folgende neue Beweis desselben Theorems nicht ohne Interesse sein, da er einerseits keine speziellen Lehrsätze der Mengentheorie voraussetzt, andererseits aber den rein formalen Charakter der Wohlordnung, die mit räumlich-zeitlicher Anordnung gar nichts zu tun hat, deutlicher als der erste Beweis hervortreten läßt.

Untersuchungen über die Grundlagen der Mengenlehre. I.

Von

E. Zermelo in Göttingen.

Die Mengenlehre ist derjenige Zweig der Mathematik, dem die Aufgabe zufällt, die Grundbegriffe der Zahl, der Anordnung und der Funktion in ihrer ursprünglichen Einfachheit mathematisch zu untersuchen und damit die logischen Grundlagen der gesamten Arithmetik und Analysis zu entwickeln; sie bildet somit einen unentbehrlichen Bestandteil der mathematischen Wissenschaft. Nun scheint aber gegenwärtig gerade diese Disziplin in ihrer ganzen Existenz bedroht durch gewisse Widersprüche oder „Antinomieen", die sich aus ihren scheinbar denknotwendig gegebenen Prinzipien herleiten lassen und bisher noch keine allseitig befriedigende Lösung gefunden haben. Angesichts namentlich der „Russellschen Anti-

MATHEMATISCHE ANNALEN 65 (1908) pp 261-281

The Axiom of Choice

***Axiom of Choice* (AC).** *Every family of nonempty sets has a choice function.*

If S is a family of sets and $\emptyset \notin S$, then a *choice function* for S is a function f on S such that

$$f(X) \in X \tag{5.1}$$

for every $X \in S$.

The Axiom of Choice postulates that for every S such that $\emptyset \notin S$ there exists a function f on S that satisfies (5.1).

If $\emptyset \in S$, we can call a choice function on $S \setminus \{\emptyset\}$ a "choice function".

$$ZFC := ZF + AC.$$

<u>Proposition</u> TFAE

(i) AC

(ii) For every X, $P(X) \setminus \{\emptyset\}$ has a choice fn.

<u>Proof</u> (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i) Suppose S is arbitrary s.t. $\emptyset \in S$.

$$X \in S \implies X \subseteq \bigcup S \implies X \in \underline{P(\bigcup S) \setminus \{\emptyset\}}$$

So if f is a choice fn for $P(\bigcup S) \setminus \{\emptyset\}$, then $f \restriction S$ is a choice fn for S.

q.e.d.

Theorem Zermelo Wellordering Theorem

(ZFC) Every set is wellorderable.

Remark. This means that in ZFC, cardinals are the canonical representatives for $\sim$-eq. classes.

So in ZFC, we can define

$$|X| := \text{the unique cardinal } \kappa \text{ s.t. } X \sim \kappa$$

Then, we get $|X| = |Y| \Longleftrightarrow X \sim Y$.

Proof. By earlier Prop, we only need to show that there is an ordinal α s.t. $\alpha \sim X$.

Let $\kappa := \aleph(X)$. Define a function from κ to $X \cup \{\text{STOP}\}$ where $\text{STOP} \notin X$ by

$$f(\alpha) := \begin{cases} c(X \setminus \operatorname{ran}(f \upharpoonright \alpha)) & \text{if this is } \neq \emptyset \\ \text{STOP} & \text{o/w} \end{cases}$$

if $c : \mathcal{P}(X) \setminus \{\emptyset\} \longrightarrow X$ is a choice function.

(*) If $\operatorname{ran}(f\restriction\alpha)$ does not contain STOP, then $f\restriction\alpha$ is an injection.

If α is least s.t. $f(\alpha) = \text{STOP}$, then $f\restriction\alpha$ is a surjection onto X

But, there has to be some α s.t. $f(\alpha) = \text{STOP}$ since otherwise

$$f : \kappa \longrightarrow X$$

is an injection by (*) in contradiction to the fact that $\kappa = \aleph(X)$.

q.e.d.

The Wellordering Theorem is equivalent to the Axiom of Choice.

Define ZWOT as an axiom and prove

$$ZF + ZWOT \vdash AC.$$

<u>Proof.</u> If X is any set, by ZWOT find relation $R \subseteq X \times X$ s.t. (X, R) is a wellorder.

$A \neq \emptyset$
$A \subseteq X$ $\quad c(A) :=$ the R-least elt. of A.

Clearly c is a choice fn for $\mathcal{P}(X)\setminus\{\emptyset\}$.

By earlier Prop., this implies AC, because X was arbitrary.

q.e.d.

CONSEQUENCES OF AC

① If X & Y are sets, then $|X| \leq |Y|$ or $|Y| \leq |X|$.

[That's obvious.]

② Ctble unions of ctble sets are ctble.

[Suppose X is a countable set and for each $x \in X$, x is countable.

There is $f : \mathbb{N} \longrightarrow X$ surjective.

For every $x \in X$ there is $g : \mathbb{N} \longrightarrow x$ surjective

$\iff S_x := \{g \,;\, g : \mathbb{N} \longrightarrow x \text{ is a surj.}\} \neq \emptyset.$

Take a choice function c for

$$\{S_x ; x \in X\}$$

So $c(S_x)$ is an element of S_x

$$\underline{g_x} : \mathbb{N} \longrightarrow x .$$

Define

$$F(n,m) := g_{f(n)}(m)$$

This is a surjection from $\mathbb{N} \times \mathbb{N}$ onto $\bigcup X$.

Now concatenate F with the usual bij. between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.

q.e.d.

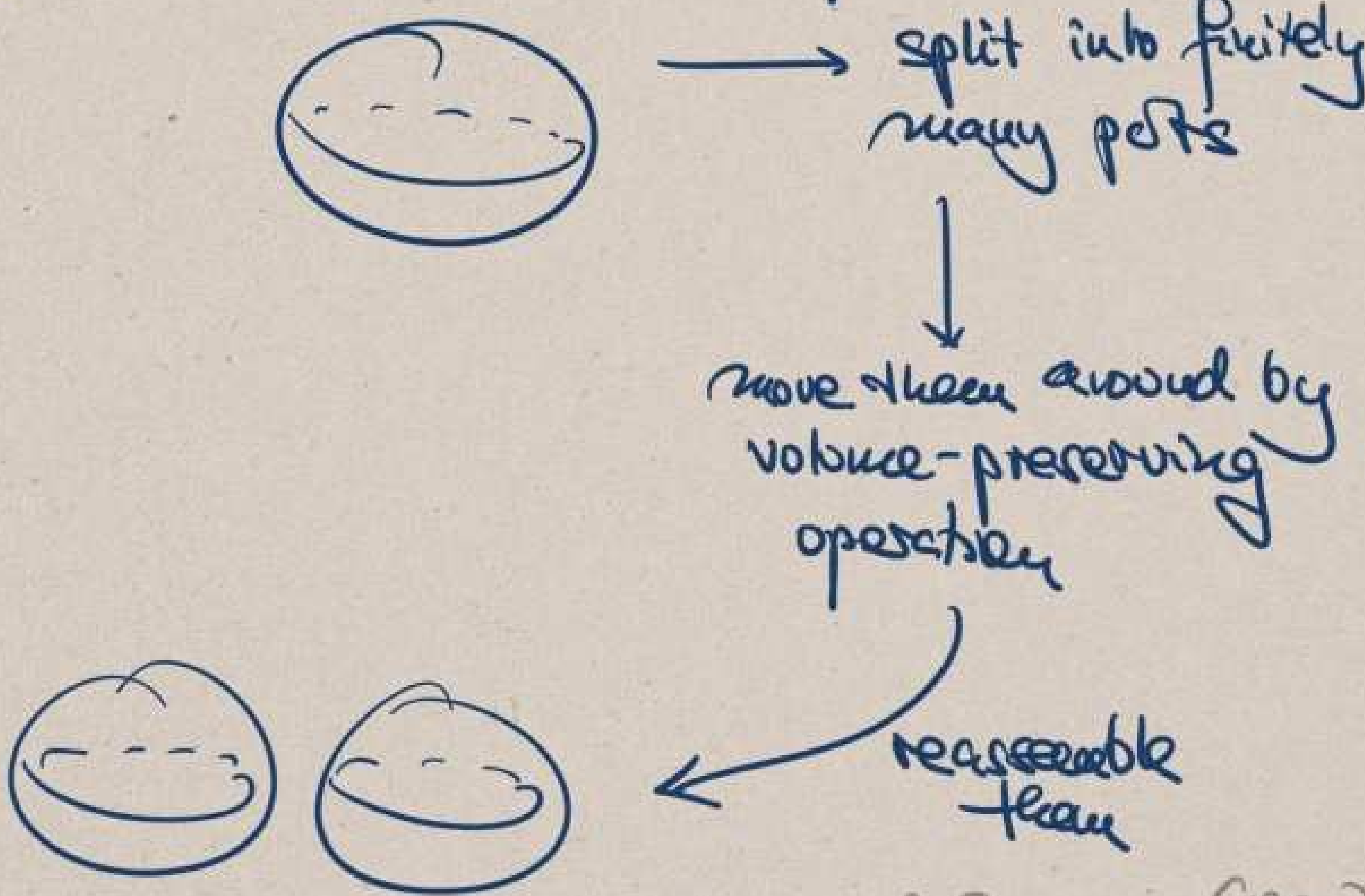
__Remark.__ It turns out that ZF cannot prove (2).

(3) Every vector space has a basis.

(4) Every field has an algebraic completion.

(5) There is a non-Lebesgue measurable set.

(6) Banach-Tarski paradox



Rubin-Rubin, Equivalents of the Axiom of Choice (1970)

Rubin-Rubin, Equivalents of the Axiom of Choice II (1985)

Howard-Rubin, Consequences of the Axiom of Choice (1998)

Some of these are equivalent to AC in the same sense as ZWOT: • (1)

- Every vector space has a basis.
- Zorn's Lemma (HW #6)

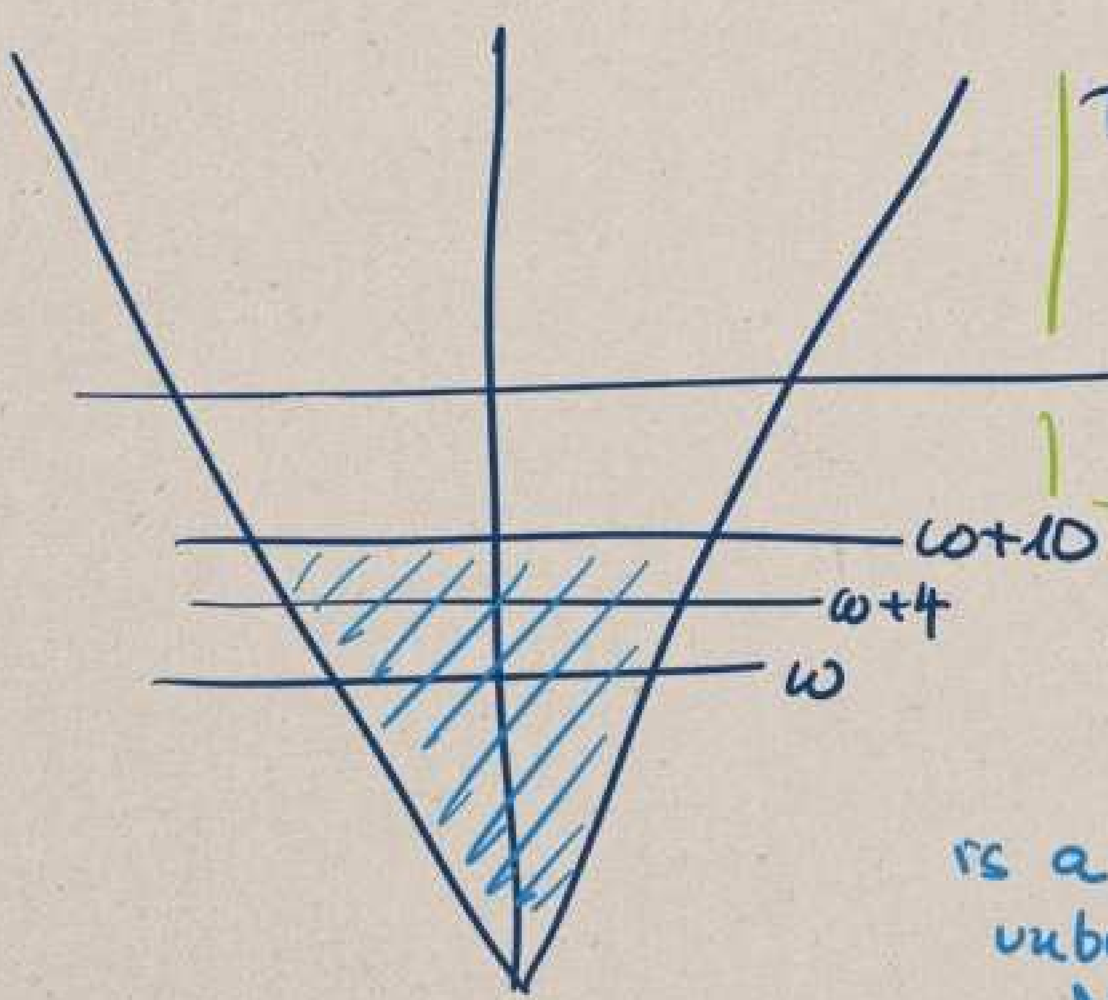
Others are so called fragments of AC: $ZF \nvdash \varphi$

$ZFC \vdash \varphi$

in general $ZF + \varphi \nvdash AC$.

Howard-Rubin studies the fragments and their relative strength.

Usually are proper fragments.



There is a non-measurable set. is bounded in the von Neumann hierarchy

Every vector space has a basis is a statement that is unbounded in the von Neumann hierarchy.

Consequences of AC for cardinals

Remember: Is there $f : \mathbb{N} \longrightarrow \omega_1$ unbounded?

What does unbounded mean:

for each $\alpha < \omega_1$ there is an $n \in \mathbb{N}$ s.t. $\alpha < f(n)$.

So, consider $\mathrm{ran}(f) \subseteq \omega_1$ a countable subset.

and for each $n \in \mathbb{N}$, $f(n)$ is a countable ordinal.

If f is unbounded, then $\bigcup \mathrm{ran}(f) = \omega_1$.

But we just proved that ctble unions of ctble sets are ctble (AC!!!), so no such unbounded fn can exist.

Remark. In ZF, this cannot be proved and it is consistent that there are unbounded fn like this.

As we saw before, limit cardinals can be small unions of smaller sets.
We'll see now that successor cardinals (with AC!!!) cannot.

Def A cardinal κ is called regular if for all $\gamma < \kappa$ and all functions

$$f : \gamma \longrightarrow \kappa \ ,$$

f is bounded.

[So, ZFC proves that ω_1 is regular.]

We saw that $f : \omega \longrightarrow \aleph_\omega$

$$f(n) := \aleph_n$$

is an unbounded function of this type, so $\aleph_\omega$ is not regular (singular).

Theorem (ZFC)

All successor cardinals are regular.

Theorem (Hessenberg)

For all $\gamma \geqslant \omega$, $\gamma \times \gamma \sim \gamma$.

[Group interaction #5.]

Proof of "every succ. is regular".

Let $\aleph_{\gamma+1}$ be our successor cardinal.

Suppose $\alpha < \aleph_{\gamma+1}$ and

$$f : \alpha \longrightarrow \aleph_{\gamma+1}.$$

There is a surjection $h : \aleph_\gamma \longrightarrow \alpha$ and for each $\beta < \alpha$, there is a surjection $g_\beta : \aleph_\gamma \longrightarrow f(\beta)$

IMPLICITLY USING A CHOICE FN FOR

$\{S_\beta \,;\, \beta < \alpha\}$

where S_β is the set of surjections

$$(\xi, \eta) \in \aleph_\gamma \times \aleph_\gamma \longmapsto g_{h(\xi)}(\eta)$$

This is a surjection from

$\aleph_\gamma \times \aleph_\gamma$ onto $\bigcup_{\beta<\alpha} f(\beta)$

By Hessenberg, $\aleph_\gamma \times \aleph_\gamma$ is in bijection with $\aleph_\gamma$,

so we get a surjection from

$\aleph_\gamma$ onto $\bigcup_{\beta<\alpha} f(\beta)$.

So f cannot be unbounded (o/w the union is $\aleph_{\gamma+1}$). q.e.d.