Set Theory

FIFTH LECTURE

11 October 2021

Ordinal Arithmetic

We shall now define addition, multiplication and exponentiation of ordinal numbers, using Transfinite Recursion.

Definition 2.18 (Addition). For all ordinal numbers α

(i) $\alpha + 0 = \alpha$,
(ii) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, for all β,
(iii) $\alpha + \beta = \underline{\lim_{\xi \to \beta}(\alpha + \xi)}$ for all limit $\beta > 0$.

$$\alpha + \beta = \bigcup_{\xi < \beta} \alpha + \xi = \bigcup \{\alpha + \xi ; \xi < \beta\} = \sup_{\xi < \beta} \alpha + \xi$$

Definition 2.19 (Multiplication). For all ordinal numbers α

(i) $\alpha \cdot 0 = 0$,
(ii) $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$ for all β,
(iii) $\alpha \cdot \beta = \lim_{\xi \to \beta} \alpha \cdot \xi$ for all limit $\beta > 0$.

DEFINITION HAVE HIGH DEGREE OF LEFT/RIGHT ACYMMETRY.

Definition 2.20 (Exponentiation). For all ordinal numbers α

(i) $\alpha^0 = 1$,
(ii) $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$ for all β,
(iii) $\alpha^\beta = \lim_{\xi \to \beta} \alpha^\xi$ for all limit $\beta > 0$.

We saw:

$1 + \underline{\omega} = \omega \neq \omega + \underline{1}$

$2 \cdot \underline{\omega} = \omega \neq \omega \cdot \underline{2}$

Theorem 2.26 (Cantor's Normal Form Theorem). *Every ordinal $\alpha > 0$ can be represented uniquely in the form*

$$\alpha = \omega^{\beta_1} \cdot \underline{k_1} + \ldots + \omega^{\beta_n} \cdot \underline{k_n},$$

where $n \geq 1$, $\alpha \geq \underline{\beta_1 > \ldots > \beta_n}$, and $k_1, \ldots, k_n$ are nonzero natural numbers.

GROUP INTERACTION #3

SWALLOWING

① If $\underline{\alpha < \beta}$, then $\omega^\alpha + \omega^\beta = \omega^\beta$.

[By subtraction (GI#3), we find $\delta > 0$ s.t. $\beta = \alpha + \delta$

$$\begin{aligned}\omega^\alpha + \omega^\beta &= \omega^\alpha + \omega^{\alpha+\delta} \\ &= \omega^\alpha + \omega^\alpha \cdot \omega^\delta \\ &= \omega^\alpha(\underbrace{1 + \omega^\delta}_{(\omega^\delta)}) \\ &= \omega^\alpha \cdot \omega^\delta = \omega^{\alpha+\delta} \\ &= \omega^\beta.\end{aligned}$$
]

② Consequence.

If $\underline{\gamma < \omega^\beta}$, then $\gamma + \omega^\beta = \omega^\beta$.

Consider CNF of γ:

$$\gamma = \omega^{\gamma_1} + \ldots + \omega^{\gamma_n} \quad \text{with } \gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n$$

since $\gamma < \omega^\beta$ $\longrightarrow$ $\gamma_1 < \beta$.

$$\underline{\leq \omega^{\gamma_1} \cdot n < \omega^{\gamma_1} \cdot \omega = \omega^{\gamma_1 + 1}} \;\Big|\; \underline{\leq \omega^\beta.}$$

$$\gamma + \omega^\beta \leq \omega^{\gamma_1} \cdot n + \omega^\beta = \omega^\beta.$$

Apply ① n times

③ Remarks about distributivity.

$\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$ holds,

$(\alpha+\beta)\gamma \underset{\text{in general}}{\neq} \alpha\gamma + \beta\gamma$.

$$\underline{(\omega+1)\cdot 2} = (\omega+1)+(\omega+1)$$
$$= \omega + \underbrace{(1+\omega)}_{=\omega} + 1$$
$$= \underline{\omega\cdot 2 + 1}$$

$$\underline{2\cdot(\omega+1)} = 2\cdot\omega + 2\cdot 1$$
$$= \underline{\omega + 2}$$

④ $(\omega^\alpha+\gamma)\cdot 2 = \omega^\alpha + \underbrace{\gamma + \omega^\alpha}_{\text{Apply ②}} + \gamma$
where $\gamma < \omega^\alpha$

$$= \omega^\alpha\cdot 2 + \gamma$$

In general, $(\omega^\alpha+\gamma)\cdot n = \boxed{\omega^\alpha\cdot n + \gamma}$

Therefore $(\omega^\alpha+\gamma)\cdot\omega = \bigcup_{n\in\mathbb{N}} \underline{(\omega^\alpha+\gamma)\cdot n}$

$$= \omega^\alpha\cdot\omega = \omega^{\alpha+1}.$$

EXAMPLE

$$\underbrace{(\omega^3 + \omega^2 \cdot 2 + 1)}_{\alpha}(\omega^2 + \omega \cdot 3 + 2)$$

$$= \alpha\omega^2 + \alpha\omega \cdot 3 + \alpha \cdot 2$$

$$\alpha \cdot 2 = (\omega^3 + \omega^2 \cdot 2 + 1)\,2 = \omega^3 + \cancel{\omega^2 \cdot 2 + 1} + \underline{\omega^3} + \omega^2 \cdot 2 + 1$$

$$= \omega^3 \cdot 2 + \omega^2 \cdot 2 + 1$$

$$\alpha \cdot \omega = (\omega^3 + \omega^2 \cdot 2 + 1) \cdot \omega = \omega^4 \quad \text{by } 4$$

$$\alpha \cdot \omega^2 = \omega^5 \quad \text{by } 4$$

$$= \omega^5 + \omega^4 \cdot \underline{3} + \omega^3 \cdot \underline{2} + \omega^2 \cdot \underline{2} + \underline{1}.$$

$$(\underline{x}^3 + \underline{2}x^2 + 1)(x^2 + 3x + 2)$$

$$= x^5 + \underline{5}x^4 + \underline{8}x^3 + \underline{5}x^2 + \underbrace{3x}_{} + \underline{2}$$

NORMAL OPERATIONS & FIXED PTS

Remember "definable operations": a functional formula Φ.

We call such a definable operation an ordinal operation if for all α ordinal there is a (unique) β ordinal s.t. $\Phi(\alpha,\beta)$.

In Jech's book, he uses symbols like

$$F(\alpha) = \beta.$$

Definition An ordinal operation F is called NORMAL if [$\Rightarrow$ f.a. α $F(\alpha) \geq \alpha$]

1. $\alpha < \beta \longrightarrow F(\alpha) < F(\beta)$
2. [CONTINUITY]
   if λ is a limit ordinal, then
   $$F(\lambda) = \bigcup \{F(\alpha); \alpha < \lambda\}$$

That's exactly what we had in ordinal arithmetic.

Examples are the operations

$$\beta \longmapsto \alpha + \beta$$
$$\beta \longmapsto \alpha \cdot \beta$$
$$\beta \longmapsto \alpha^\beta.$$

All of them are normal.

Def. If F is an ordinal operation, α is a fixed pt of F if $F(\alpha)=\alpha$.

Thm If F is a normal ordinal operation and α is an ordinal, then there is some $\gamma>\alpha$ s.t. γ is a fixed pt. for F.

Proof. Define the following sequence by recursion:

$$\gamma_0 := \alpha+1.$$
$$\gamma_{n+1} := F(\gamma_n).$$

Needs Rec. Thm w/o fixed range.

Case 1. One of the γ_n is a fixed pt. Done.

Case 2. Not. That means γ_n is a strictly increasing seq. of ordinals. Therefore $\gamma := \bigcup\{\gamma_n ; n\in\mathbb{N}\}$ is a limit ordinal.

$$F(\gamma) \underset{\text{by continuity}}{=} \bigcup\{F(\alpha);\alpha<\gamma\}$$
$$= \bigcup\{F(\gamma_n);n\in\mathbb{N}\}$$
$$= \bigcup\{\gamma_{n+1};n\in\mathbb{N}\} = \gamma.$$

q.e.d.

Illustration

Take function $\beta \longmapsto \omega + \beta$.
This is normal, therefore has fixed pts.
What is the smallest fixed pt?

$$\gamma_0 = 0$$
$$\gamma_1 = \omega + 0 = \omega \cdot 1$$
$$\gamma_2 = \omega + (\omega + 0) = \omega \cdot 2$$
$$\gamma_3 = \omega + \gamma_2 = \omega \cdot 3$$
$$\gamma_n = \omega \cdot n$$

$$\bigcup \{\gamma_n ; n \in \mathbb{N}\} = \omega \cdot \omega = \omega^2.$$

[This is in line with (1) and (2) above since ω^2 is a fixed pt for all operations $\beta \longmapsto \gamma + \beta$ where $\gamma < \omega^2$.]

Remark. These fixed pts are closely related to the questions on gamma numbers and delta numbers on GI#3.

Back to our original motivation:

- We wanted ordinals as canonical representatives from the isomorphism classes of wellorders.
- We have proved that if α, β ordinals and $(\alpha, \in) \cong (\beta, \in)$, then $\alpha = \beta$.
- We still need to show that each isomorphism class contains an ordinal.

<u>Theorem</u> (ZF$_0$) If $(X, <)$ is a wellorder, there is an ordinal α s.t.
unique

$$(X, <) \cong (\alpha, \in).$$

REPRESENTATION THEOREM FOR WELLORDERS

<u>Proof.</u> We define by recursion a function on X:

$$F(x) = \operatorname{ran}(F \upharpoonright <[x])$$

Claim 1 F is order-preserving between $<$ and $\in$

If $x<y$, then $F(x) \in F(y)$

[By recursion equation.]

Define $A := \mathrm{ran}(F)$.

Claim 2 A is an ordinal.

A is well ordered by $\in$ [follows from Claim 1]

A is transitive:

$$v \in \underset{\overset{\|}{F(x)}}{w} \in A$$

by rec. eq. $v = F(y)$ for some $y<x$

$\Longrightarrow v \in A$.

So $(X,<) \cong (A,\in)$. q.e.d.

The fact that there is no set of all ordinals means that INTUITIVELY there must be lots of them.

But: are there even uncountable ordinals?

RECURSION EQUATION

$F(x) = \mathrm{ran}(F \upharpoonright {<}[x])$

Theorem <u>HARTOGS'S THEOREM</u>

If X is any set, then there is an ordinal α s.t. there is no injection from α into X.

[Consequence: if $X := \mathbb{N}$, then α is an uncountable ordinal.]

Proof [Idea: Consider A_X, the collection of all ordinals α s.t. there is an inj. from α to X.
Show that A_X is a set. Show that A_X is transitive. So A_X is an ordinal: $A_X \notin A_X$.]

Consider
$$W_X := \{(A,R) ; A \subseteq X \text{ and } (A,R) \text{ is a wellorder}\}$$

This is a set by Separation and the usual operations $\mathcal{P}$, $\times$, ...

By Representation Thm, each $(A,R) \in W_X$ corresponds to a unique ordinal α s.t.
$$(A,R) \cong (\alpha, \in).$$

This is a functional formula, so by Replacement

$$\hat{W}_X := \{ \alpha \,;\, \exists (A,R) \in W_X \text{ s.t. } (\alpha,\in) \cong (A,R) \}$$

is a set.

<u>Claim</u> $\alpha \in A_X \iff \alpha \in \hat{W}_X$

"$\Leftarrow$" The isomorphism $(\alpha,\in) \cong (A,R)$ is an injection from α to X.

"$\Rightarrow$": Suppose $f : \alpha \longrightarrow X$ is an injection.

$A := \mathrm{ran}(f)$, so $f : \alpha \longrightarrow A$ is a bijection.

Define $R \subseteq A \times A$ by

$$f(\beta) R f(\gamma) :\iff \beta \in \gamma$$

Then f is an isom. between $(\alpha,\in)$ and (A,R)

"Coding".

$$\Longrightarrow \alpha \in \hat{W}_X .$$

q.e.d.

<u>Claim</u> A_X is transitive.

[$\alpha \in \beta \in A_X \overset{\text{T.S.}}{\Longrightarrow} \alpha \in A_X$. If $\alpha \in \beta \Rightarrow \alpha \subseteq \beta \Rightarrow$ if $\beta \in A_X$, then $\alpha \in A_X$.]

Remark.

This is a constructive proof. It provides a concrete example A_X of an ordinal that doesn't inject into X and it is the least example:

if γ is any ordinal:

1. $\gamma \in A_X \Longrightarrow \gamma$ injects into X
2. $\gamma = A_X$
3. $A_X \in \gamma \Longrightarrow A_X \subseteq \gamma$ $\Longrightarrow \gamma$ does not inject into X.

Concretely: The set of all countable ordinals is the least uncountable ordinal.

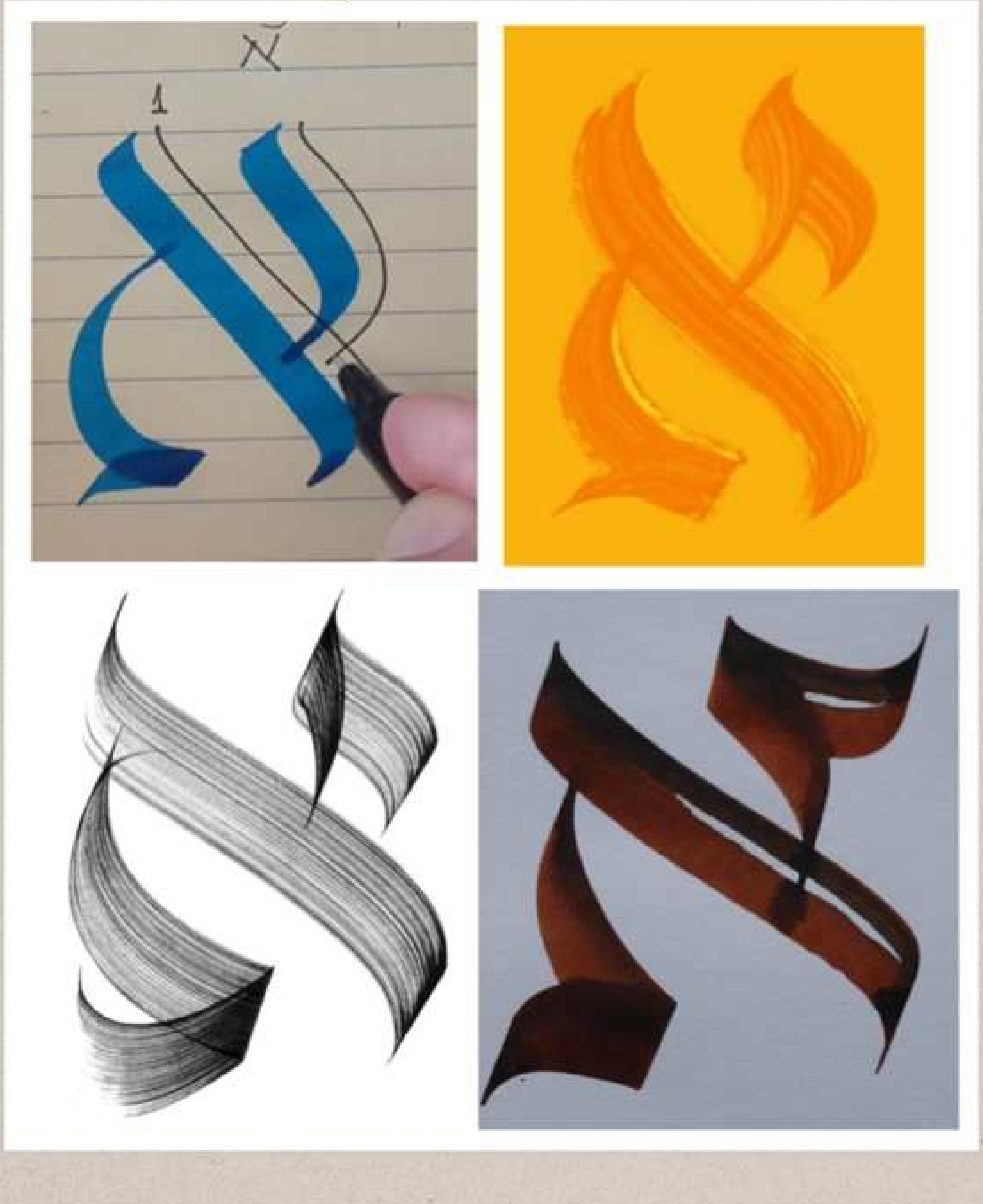
Definition This ordinal is called the Hartogs-Aleph of X, in symbols,

$$\aleph(X) := \{\alpha ;\ \alpha \text{ injects into } X\}$$

The first uncountable ordinal

$$\omega_1 := \aleph_1 := \aleph(\mathbb{N})$$

After that $\omega_2 := \aleph_2 := \aleph(\aleph_1)$



The last missing ZF axiom:

FOUNDATION

[Jech: REGULARITY]

$$\mathrm{FST} \subseteq \mathrm{Z} \subseteq \mathrm{ZF}_0$$

FST+Inf Z+Repl.

There is no intrinsic argument for Foundation, but it makes the theory of models of set theory much nicer.

Another point: so far, we have not yet excluded the possibility of a set containing itself as an element.

ANSC Axiom of no self-containment

$$\forall x\,(x \notin x)$$

Note that

 $\models$ ANSC,

but we have $\exists x \exists y\,(x \in y \wedge y \in x)$

The Cumulative Hierarchy of Sets

VON NEUMANN HIERARCHY.

We define, by transfinite induction,

$$\boxed{\begin{aligned} &V_0 = \emptyset, \qquad V_{\alpha+1} = P(V_\alpha), \\ &V_\alpha = \bigcup_{\beta<\alpha} V_\beta \quad \text{if } \alpha \text{ is a limit ordinal.}\end{aligned}}$$

The sets V_α have the following properties (by induction):

(i) Each V_α is transitive.
(ii) If $\alpha < \beta$, then $V_\alpha \subset V_\beta$.
(iii) $\alpha \subset V_\alpha$.

Proof (i): V_0 is transitive. & λ is limit
If V_α is transitive for $\alpha<\lambda$, then V_λ is transitive.

To Show: If V_α is transitive, then so is $P(V_\alpha)$.

$$\frac{x \in y \in P(V_\alpha)}{y \subseteq V_\alpha} \Longrightarrow x \in V_\alpha$$

$$\Big\Downarrow \; V_\alpha \text{ transitive}$$

$$x \in P(V_\alpha) \Longleftarrow x \subseteq V_\alpha$$

Proof (ii) Transfinite ind. on β.
If $\beta = 0$: nothing to show.
If β limit, then the claim follows from definition.
The successor case reduces to $\beta = \gamma+1$ and $V_\gamma \subseteq V_{\gamma+1}$. Suppose $x \in V_\gamma \overset{}{\underset{\text{trans.}}{\Longrightarrow}} x \subseteq V_\gamma$
$\Longrightarrow x \in V_{\gamma+1}$.

Strengthening of (iii):

$$O_\alpha := \{\beta \in V_\alpha ; \beta \text{ is an ordinal}\}$$

Claim $O_\alpha = \alpha$.

Proof by induction $\alpha = 0$. Clear.

If β is a limit,

$$O_\beta = \bigcup\{O_\alpha ; \alpha < \beta\} \overset{IH}{=} \bigcup\{\alpha ; \alpha < \beta\} = \beta.$$

Suppose $O_\alpha = \alpha$. Show that $O_{\alpha+1} = \alpha+1$.

① Show that $\alpha \in O_{\alpha+1}$.
We have by IH, $\alpha \subseteq V_\alpha$, so $\alpha \in P(V_\alpha) = V_{\alpha+1}$.

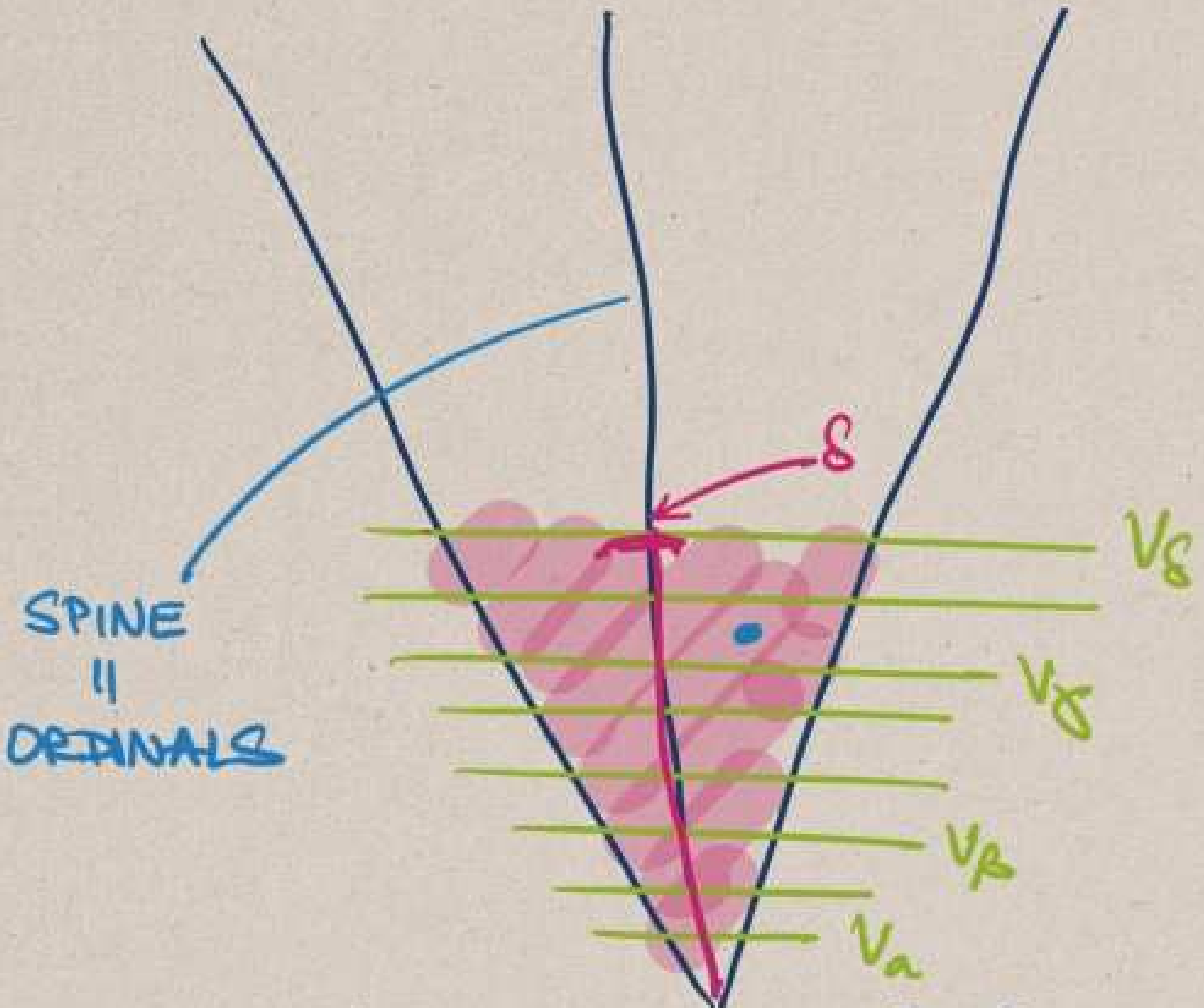
② Show that $\alpha+1 \notin O_{\alpha+1}$.
Otherwise $\alpha+1 \in V_{\alpha+1}$

$\Longrightarrow \alpha+1 \subseteq V_\alpha$

$\Longrightarrow \alpha \in \alpha+1 \subseteq V_\alpha$

$\Longrightarrow \alpha \in V_\alpha$ in contradiction to $O_\alpha = \alpha$.

q.e.d.



This stratified picture of the cumulative hierarchy allows us to define a rank function (MIRIMANOFF rank)

if x is in the cumulative hierarchy (i.e., $\exists\alpha\ x\in V_\alpha$)

$$\rho(x) := \alpha \iff x \in V_{\alpha+1}\setminus V_\alpha$$

This gives us the opportunity to prove properties in the cumulative hierarchy by induction on the Mirimanoff rank.

AXIOM OF FOUNDATION / REGULARITY.

$$\forall x \left(x \neq \emptyset \longrightarrow \exists m \left(m \in x \wedge m \cap x = \emptyset \right) \right)$$

FST +

Foundation implies ANSC:

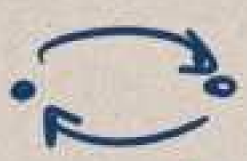
if $x \in x$, form $y := \{x\}$.

if m is any element of y $[m = x]$,

then $m \cap y \ni x$

$\Longrightarrow m \cap y \neq \emptyset$.

It also excludes:



and the existence of an infinite descending chain: f with $\mathrm{dom}(f) = \mathbb{N}$ s.t. $f(n+1) \in f(n)$.

$$ZF = ZF_0 + \text{Foundation}.$$

Theorem ($\in$-induction)

(ZF) If φ is any formula s.t.

for all x

if $\forall y \, (y \in x \longrightarrow \varphi(y))$

$\longrightarrow \varphi(x)$

then φ holds for all x.

Proof. If x is any set, let us define by recursion

$$f(0) := x$$
$$f(n+1) := \bigcup f(n)$$

By Rec. Thm. w/o range we get a function f with $\mathrm{dom}(f) = \mathbb{N}$ and can form $\bigcup_{n \in \mathbb{N}} f(n) =: TC(x)$

TRANSITIVE CLOSURE OF X

Then $TC(x)$ is transitive. and $x \subseteq TC(x)$.

[In ZF_0, every set is contained in a transitive set.]

Suppose $\in$-induction is false and derive a contradiction.

(*) for any x if $\forall y\,(y \in x \longrightarrow \varphi(y)) \longrightarrow \varphi(x)$

But there is some x_0 s.t. $\varphi(x_0)$ fails.

<u>Case 1.</u> For all $y \in x_0$, $\varphi(y)$ holds.

This contradicts (*).

<u>Case 2.</u> For some $y \in x_0$, $\varphi(y)$ doesn't hold.

$$\{y \in x_0 ;\ \neg\varphi(y)\} \neq \emptyset.$$

Let $T := TC(x_0)$.

$$W := \{y \in T ;\ \neg\varphi(y)\} \neq \emptyset.$$

By Foundation, find m minimal in W, i.e., $m \in W$ and $\underbrace{m \cap W = \emptyset}$.

$$\{z \in m ;\ z \in W\} = \emptyset$$

$$\{z \in m ;\ z \notin W\} = m$$

$\Longrightarrow$ for all $z \in m$ we have $\varphi(z)$

$\xRightarrow{(*)} \varphi(m)$. Contradiction.

q.e.d.

Corollary (ZF)

$$\forall x \, \exists \alpha \quad x \in V_\alpha .$$

Proof. Suppose x is an arbitrary set. And suppose for all $y \in x$ there is α s.t. $y \in V_\alpha$.

By Replacement, form

$$\{\rho(y)+1 ; y \in x\}$$

and $\gamma := \bigcup \{\rho(y)^{+1} ; y \in x\}$

is an ordinal.

Then if $y \in x$, $y \in V_\gamma$.

Thus $x \subseteq V_\gamma \implies x \in V_{\gamma+1}$.

q.e.d.

Thus, Foundation implies that the von Neumann hierarchy is the set-theoretic universe.

Consistency proofs with the von Neumann hierarchy

Most axioms are $\boxed{\forall\exists}$

So showing that $V_\alpha \models \Phi$ is about analysing what the axiom does to the Mirimanoff rank:

Ex. x, y $\rho(x) = \alpha$ w.l.o.g.
$\rho(y) = \beta$ $\alpha \leq \beta$

$$\rho(\{x, y\}) = \beta + 1.$$

So, if γ is closed under

$$\alpha, \beta \longmapsto \max(\alpha, \beta) + 1,$$

then $V_\gamma \models$ Pair.

Similarly for Union: $\rho(x) = \alpha$
then $\rho(\bigcup x) \leq \alpha$.

Powerset: $\rho(x) = \alpha$
then $\rho(\mathcal{P}(x)) = \alpha + 1$

Separation If $\rho(x) = \alpha$, then
$\rho(\{y \in x ; \varphi(y)\}) = \alpha$.

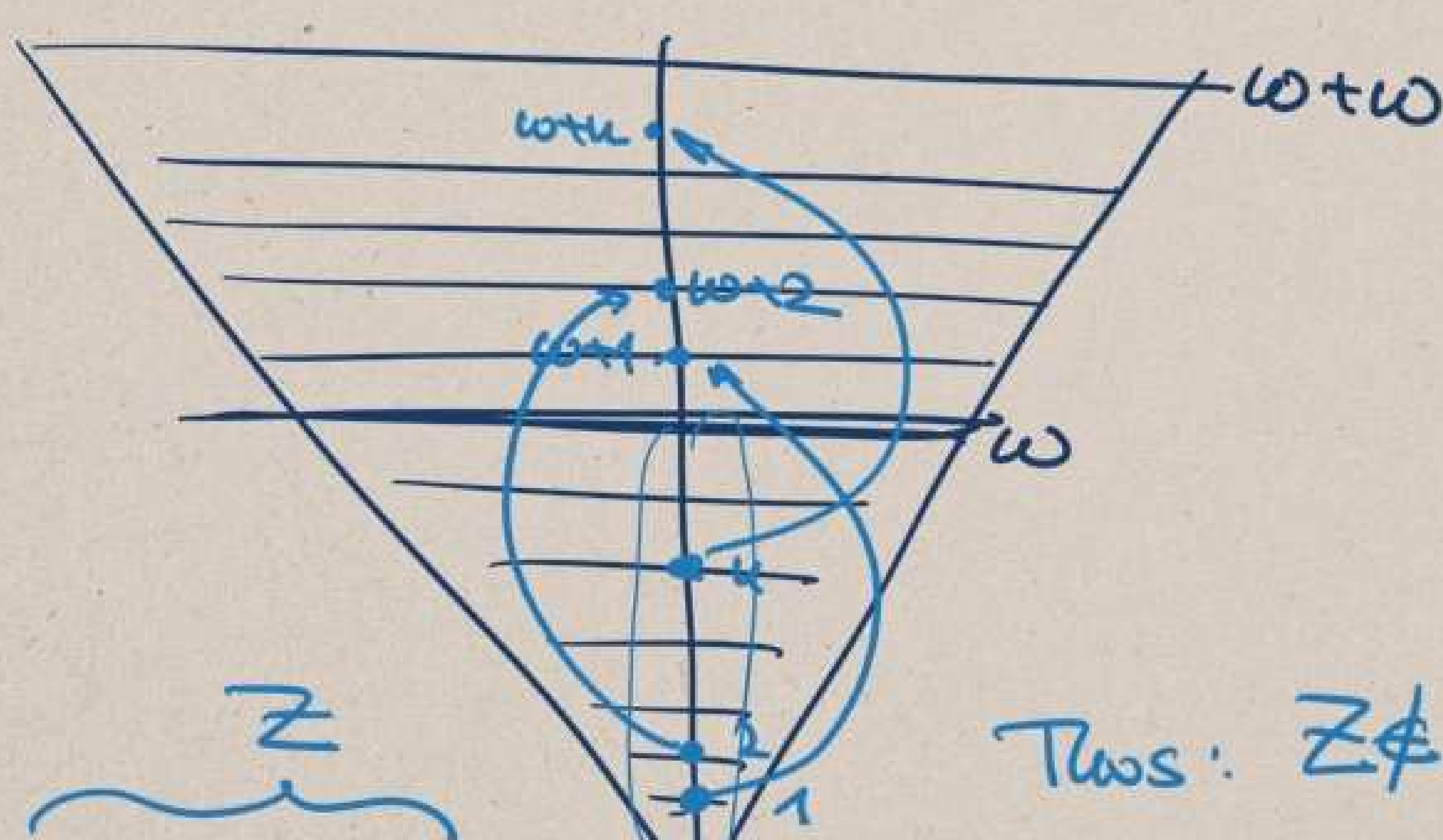
Together

If γ is limit, then $V_\gamma \models$ FST.

If $\underbrace{\omega \in V_\gamma}_{\substack{\Longleftrightarrow \\ \gamma \geqslant \omega+1}}$, then $V_\gamma \models$ Inf.

Together

If $\gamma > \omega$ is a limit, then $V_\gamma \models$ Z.



$\overbrace{\rho(\{\omega+n; n\in\mathbb{N}\})}^{Z} = \omega+\omega$

Thus: $Z \notin V_{\omega+\omega}$.