Set Theory — IV

4 October 2021

PRINCIPLE OF COMPLETE INDUCTION

$(X, s, 0) \leftarrow$

$Z \subseteq X : 0 \in Z \wedge$
$\forall x \, (x \in Z \rightarrow s(x) \in Z)$
$\Longrightarrow Z = X$

RECURSION THEOREM

RECURSION PRINCIPLE

$f : Y \longrightarrow Y, \; y_0 \in Y$

There is a unique $F : X \longrightarrow Y$ s.t.

$$\boxed{\begin{aligned} F(0) &= y_0 \\ F(s(x)) &= f(F(x)) \end{aligned}}$$

PRINCIPLE OF ORDER INDUCTION

$(X, <) \leftarrow$

$Z \subseteq X : \forall x$
if $<[x] \subseteq Z$, then $x \in Z$
$\Longrightarrow Z = X$.

RECURSION THEOREM

ORDERTHEORETIC RECURSION PRINCIPLE

$P := \{ g \subseteq X \times Y \, ; \, \exists x \; \mathrm{dom}(g) = <[x] \}$

$f : P \longrightarrow Y$

There is a unique $F : X \longrightarrow Y$ s.t.

$$\boxed{F(x) = f(F \upharpoonright <[x])}$$

Example : $(\mathbb{N}, s, 0)$ $\quad (\mathbb{N}, <)$

Def. A strict total order $(X, <)$ is called a wellorder if it satisfies the least number principle

($\iff$ $<$ is wellfounded

$\iff$ every nonempty subset $Z \subseteq X$, $Z \neq \emptyset$ has a minimal element).

HW#3 shows: wellorders satisfy the principle of order induction.

Example $(\mathbb{N}, <)$

$$s(\mathbb{N}) = \boxed{\mathbb{N} \cup \{\mathbb{N}\}}$$

$\boxed{(s(\mathbb{N}), \in)}$ is a wellorder

HW:

$$(s(\mathbb{N}), \in) \cong (\mathbb{N}, <) \oplus (1, <)$$

By homework, this is once more a wellorder.

The operation $\oplus$ gives us even longer wellorders:

$(\mathbb{N},<) \oplus (\mathbb{N},<)$

Explicit coding of that wellorder:

Define $<_{EO}$ on $\mathbb{N}$ by

$$n <_{EO} m \;:\Longleftrightarrow \begin{pmatrix} n \,\&\, m \text{ are even} \\ \text{and } n<m \end{pmatrix} \text{ OR } \begin{pmatrix} n \,\&\, m \text{ are odd} \\ \text{and } n<m \end{pmatrix} \text{ OR } \begin{pmatrix} n \text{ is even and} \\ m \text{ is odd} \end{pmatrix}$$

Then $(\mathbb{N},<) \oplus (\mathbb{N},<) \cong (\mathbb{N},<_{EO})$

__Remark__ There is also a product operation preserving wellorders! HW #4.

Properties of wellorders

Remember if $(X,<)$ is a strict total order, then for $z\in X$, $\boxed{<[z]} := \{y\in X;\ y<z\}$

Def. $I\subseteq X$ is called *initial segment* if for all $x,y\in X$ if $x<y$ and $y\in I$, then $x\in I$.

I is called *proper initial segment* if I is initial segment & $I\neq X$.

The sets $<[z]$ for $z\in X$ are always proper initial segments.

Proposition If $(X,<)$ is a wellorder, then every proper initial segment is of the form $<[z]$ for some $z\in X$.

[Remark. This means that there is a 1-to-1 correspondence between proper initial segments of X and elements of X.]

Proof of Prop.

Let $I \subseteq X$ be a proper initial segment.

$$I \subsetneq X,$$

so $X \setminus I \subseteq X$ and $X \setminus I \neq \emptyset$.

By wellfoundedness of $<$, we find a minimal elt. $z \in X \setminus I$.

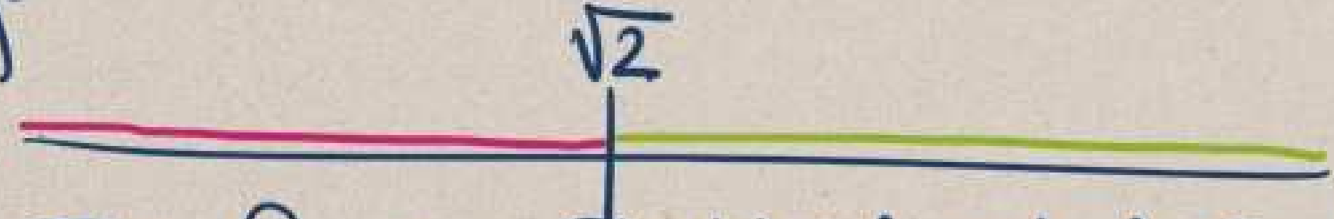
Claim: $\quad I = {<}[z]$.

"$\subseteq$" because I is initial segm.

"$\supseteq$" because z is minimal.

q.e.d.

Remark Note that this is not the case for non-wellorders.



This forms a Dedekind cut of $\mathbb{Q}$, $\mathbb{Q}$

$I := \{q \in \mathbb{Q} ; q < \sqrt{2}\}$ is a proper initial segment; but for each $q_0 \in \mathbb{Q}$,

$$I \neq {<}[q_0].$$

Jech

§2

Lemma 2.4. *If $(W,<)$ is a well-ordered set and $f : W \to W$ is an increasing function, then $f(x) \geq x$ for each $x \in W$.*

RIGID

Corollary 2.5. *The only automorphism of a well-ordered set is the identity.*

Corollary 2.6. *If two well-ordered sets W_1, W_2 are isomorphic, then the isomorphism of W_1 onto W_2 is unique.* □

Lemma 2.7. *No well-ordered set is isomorphic to an initial segment of itself.*

Proof L 2.4

Towards a contradiction, let's assume

$$X := \{x \in W ;\ f(x) \boxed{\ngeq} x\} \neq \emptyset$$

$<$ (because $<$ is a strict total order)

By the least number principle, find $x_0 \in X$ minimal:

$$f(x_0) < x_0 \qquad (*)$$

Monotonicity of f:

$$f(\boxed{f(x_0)}) < \boxed{f(x_0)}$$

So $f(x_0) \in X$, contradicting by $(*)$ the minimality of x_0.

q.e.d.

Proof C 2.5

If $f : W \to W$ is an automorphism, then both f and f^{-1} satisfy L 2.4:

$$f(x) \geq x$$
$$f^{-1}(x) \geq x \Rightarrow x \geq f(x)$$
$$\Rightarrow x = f(x).$$

q.e.d.

<u>Proof C 2.6</u> If $f_1 : \underline{W_1} \longrightarrow \underline{W_2}$ iso
and $f_2 : \underline{W_1} \longrightarrow \underline{W_2}$ iso

Claim: $f_1 = f_2$.

So $f_2^{-1} \circ f_1$ is an automorphism of W_1,
therefore $f_2^{-1} \circ f_1 = id$.

Similarly, $f_2 \circ f_1^{-1} = id$.

So f_1 and f_2 are the same. q.e.d.

<u>Proof L 2.7</u> Suppose we have a iso

$$f : \underline{W} \longrightarrow \underline{I}$$

where $\underline{I}$ is a proper initial segment of W.
By the previous proposition, we know
that $\underline{I} = <[x]$ for some $\underline{x \in W}$.

Then $f(x) \in \underline{I} = <[x]$

$$\Longrightarrow f(x) < x$$

in contradiction to L 2.4.

q.e.d.

__Remark.__ L 2.7 is not true for non-wellorders.

$\mathbb{Q}$ and $<[0]$

are isomorphic.

L 2.7 allows us to define an ordering on wellorders:

$$(W_1,<_1) < (W_2,<_2)$$

iff $(W_1,<_1)$ is isomorphic to a proper initial segment of $(W_2,<_2)$.

This relation is obviously transitive, and L 2.7 says that it is irreflexive.

__Next goal__: Totality of $<$.

Jech

FUNDAMENTAL THEOREM ON WELLORDERS.

Theorem 2.8. *If W_1 and W_2 are well-ordered sets, then exactly one of the following three cases holds:*

(i) *W_1 is isomorphic to W_2;*
(ii) *W_1 is isomorphic to an initial segment of W_2;* $W_1 \prec W_2$
(iii) *W_2 is isomorphic to an initial segment of W_1.* $W_2 \prec W_1$

[This is almost the totality of $\prec$, except that option (i) is not $W_1 = W_2$, but "is isomorphic to". So, it's totality up to isomorphism.]

Proof. Define a relation

$$f \subseteq W_1 \times W_2 \text{ as follows}$$

$$(x,y) \in f :\Longleftrightarrow <_1[x] \cong <_2[y].$$

We claim:

(1) f is functional: $<_1[x] \cong <_2[y]$
$<_1[x] \cong <_2[y']$

$$\Downarrow$$

$$<_2[y] \cong <_2[y']$$

But if $y \neq y'$, w.l.o.g. $y <_2 y'$ and therefore $<_2[y]$ is a proper initial segment of $<_2[y']$, contradicting L2.7.

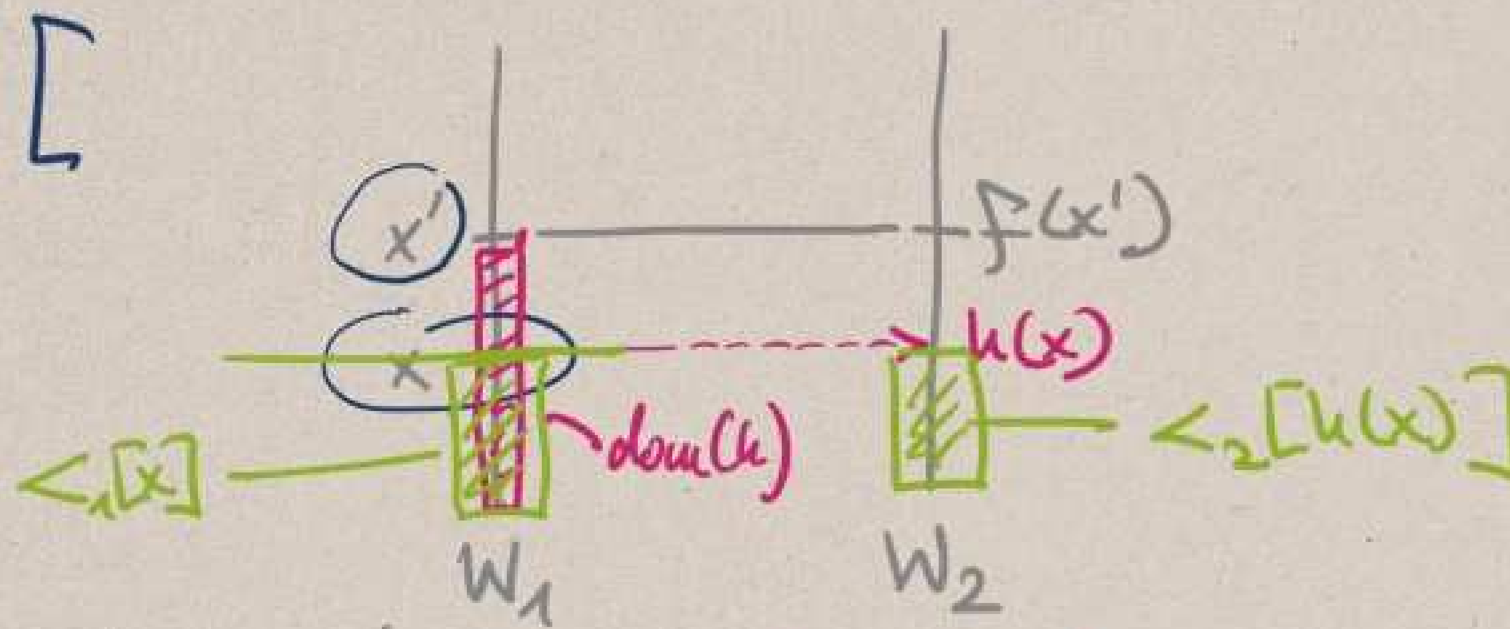
(2) f is injective

[Exactly the same proof.]

(3) f is orderpreserving

$$x <_1 x' \Longrightarrow f(x) <_2 f(x').$$

[



By definition of f, we have iso

$$\underline{h} : \underline{<_1[x']} \cong \underline{<_2[f(x')]}$$

Since $x <_1 x'$, $x \in \mathrm{dom}(h)$ and

$$\boxed{h(x) <_2 f(x')}.$$

Clearly $h \restriction <_1[x]$ is an isomorphism between $<_1[x]$ and $<_2[h(x)]$

But by def. of f, this means $\boxed{h(x) = f(x)}$

$$\Longrightarrow f(x) <_2 f(x').$$]

(4) $\mathrm{dom}(f)$ is an initial segment of W_1

(5) $\mathrm{ran}(f)$ is an initial segment of W_2

[(4) is precisely what the previous proof showed; (5) uses h^{-1}.]

Four cases

dom \ ran	proper	not proper
proper	$\times$	$f: <_1[x] \longrightarrow W_2$ $W_1 < W_2$ Case (ii)
not proper	$W_2 < W_1$ Case (iii)	$W_1 \cong W_2$ Case (i)

Left to show that not both can be proper:

$$f: <_1[x] \longrightarrow <_2[y]$$

By def. $(x,y) \in f$ in contradiction to $\mathrm{dom}(f) = <_1[x]$.

$\Rightarrow \times$

q.e.d.

Goal Find canonical representatives in the isomorphism classes of wellorders.

THE ORDINALS

Def. A set α is called an **ORDINAL** if it is transitive ("elements are subsets") and $(\alpha, \in)$ is a wellorder.

Examples (1) Our work on $\mathbb{N}$ shows:

(a) each $n \in \mathbb{N}$ is an ordinal

(b) $\mathbb{N}$ itself is an ordinal.

(2) If α is an ordinal, then so is $s(\alpha) = \alpha \cup \{\alpha\}$

[$(s(\alpha), \in) \cong (\alpha, \in) \oplus (1, \in)$, so by the results on order sums, this is a wellorder.

Claim: $s(\alpha)$ is transitive.

This follows from the general statement "if x trs $\Rightarrow s(x)$ trs." as proved in the section on $\mathbb{N}$]

Lemma 2.11.

Jech

(i) $0 = \emptyset$ *is an ordinal.*
(ii) *If* α *is an ordinal and* $\beta \in \alpha$, *then* β *is an ordinal.*
(iii) *If* $\alpha \neq \beta$ *are ordinals and* $\alpha \subset \beta$, *then* $\alpha \in \beta$.
(iv) *If* α, β *are ordinals, then either* $\alpha \subseteq \beta$ *or* $\beta \subseteq \alpha$.

Jech's $\subset$ means $\subsetneq$

(i) ✓

(ii) Observe that if $(X, <)$ is a wellorder and $Y \subseteq X$, then $(Y, <) = (Y, < \cap\, Y \times Y)$ is a wellorder.

Since α is transitive, every element of α is a subset of α, thus

$(\beta, \in)$ is a wellorder

since $\beta \in \alpha \Longrightarrow \beta \subseteq \alpha$.

The transitivity of the set β follows from the transitivity of the relation $\in$ on α.

(iii) Suppose $\alpha \subsetneq \beta$, that means $\beta \setminus \alpha \neq \emptyset$

Find γ minimal in $\beta \setminus \alpha$.

What is the proper initial segment def'ed by γ: $\in[\gamma] = \{\delta \in \beta ; \delta \in \gamma\}$

$= \gamma$.

So $\gamma = \alpha$. But $\gamma \in \beta$, so $\alpha \in \beta$.

(iv) If α, β ordinals, then $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

[Define $\gamma := \alpha \cap \beta$. This is an ordinal since intersections of transitive sets are transitive & $\gamma \subseteq \alpha$, so $(\gamma, \in)$ is a wellorder.

$$\boxed{\gamma \subseteq \alpha \quad \text{and} \quad \gamma \subseteq \beta}$$

$$\left.\begin{array}{l}\text{If } \gamma = \alpha, \text{ then } \alpha \subseteq \beta \\ \phantom{\text{If }} \gamma = \beta, \text{ then } \beta \subseteq \alpha\end{array}\right\} \text{Done!}$$

So, the remaining case is:

$$\gamma \subsetneq \alpha \quad \text{and} \quad \gamma \subsetneq \beta$$

$$\text{(iii)} \Downarrow \qquad\qquad \Downarrow \text{(iii)}$$

$$\gamma \in \alpha \qquad\qquad \gamma \in \beta$$

$$\Rightarrow \gamma \in \alpha \cap \beta = \gamma$$

Since $\gamma \in \alpha$, this contradicts irreflexivity of $\in$ on α.

q.e.d.]

(iii) & (iv) show that ordinals satisfy totality of $\in$:

If α, β are ordinal, then

either $\alpha = \beta$

or $\alpha \subsetneqq \beta \overset{(iii)}{\Longrightarrow} \alpha \in \beta$

or $\beta \subsetneqq \alpha \overset{(iii)}{\Longrightarrow} \beta \in \alpha$.

Also, the $\in$ relation is wellfounded on ordinals in the following sense:

If Z is a non-empty set of ordinals, then Z has a minimal element.

[$Z \neq \emptyset$, let $\xi \in Z$.

<u>Case 1.</u> ξ is minimal. Done.

<u>Case 2.</u> ξ is not minimal. This means that

$$\xi \cap Z \neq \emptyset.$$

But $\xi \cap Z \subseteq \xi$. Apply wellfoundness of ξ to get a least elt of $\xi \cap Z$. This is minimal in Z.]

Corollary There is no set of all ordinals.

Proof. Suppose there is, call it Ω.

L 2.11 (ii) shows that Ω is a transitive set.

And the previous argument shows that $\in$ is a strict total order that is wellfounded: $(\Omega, \in)$ is a wellorder.

So Ω is an ordinal.

So $\Omega \in \Omega$, so $\in$ is not irreflexive on Ω.

Contradiction! q.e.d.

Intuitively, results like the last corollary suggest

THERE ARE LOTS OF ORDINALS

Is this actually true?

Let's build ordinals:

0

1

2

3

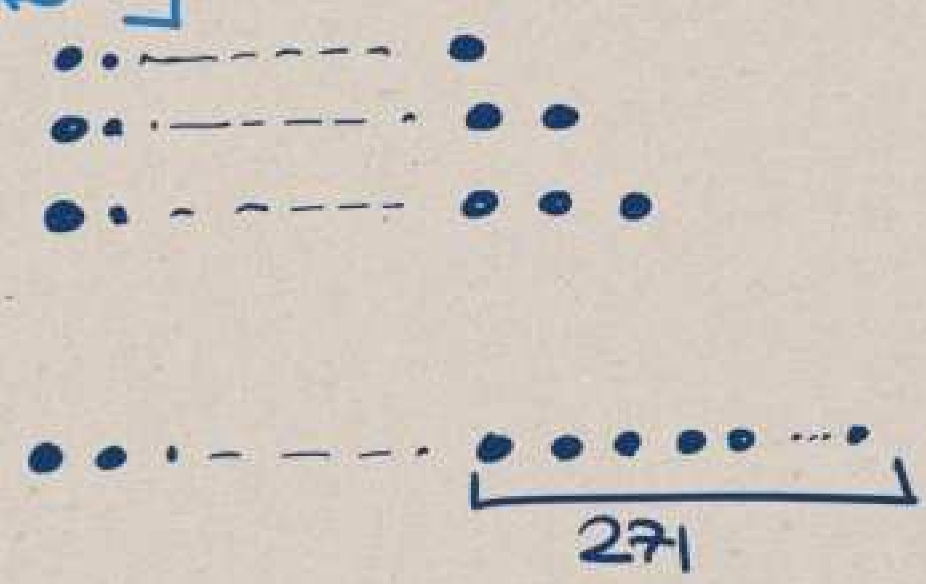
$\vdots$

$\omega := \mathbb{N}$ [ω is the name of $\mathbb{N}$ if we wish to emphasize its "ordinal character"]

$\omega+1 \sim s(\mathbb{N})$

$\omega+2 \sim s(s(\mathbb{N}))$

$\omega+3 \sim s(s(s(\mathbb{N})))$



$\vdots$

$\omega+271 \sim \underbrace{s\cdots s}_{271 \text{ times}}(\mathbb{N})$

The obvious "ordinal" bigger than all of these should be:

$\mathbb{N} \cup \{\omega+n ; n \in \mathbb{N}\}$

Can we prove that this exists in Z?

One idea:

Define by recursion

$$\underline{g}(0) := \omega$$

$$\underline{g}(n+1) := s(\underline{g}(n))$$

Apply Rec Theorem to get unique function $\underline{g}$ with $\mathrm{dom}(\underline{g}) = \mathbb{N}$ and then

$$\mathrm{ran}(\underline{g}) = \{\omega+n \,;\, n \in \mathbb{N}\}$$

THIS DOES NOT WORK!

OUR RECURSION THM SO FAR REQUIRES A FIXED RANGE FROM WHICH WE SEPARATE.

BUT HERE: THIS RANGE IS THE SET WHOSE EXISTENCE WE'RE TRYING TO PROVE!

From page 1 :

ORDERTHEORETIC RECURSION PRINCIPLE

$P := \{g \subseteq X \times Y \, ; \, \exists x \ \mathrm{dom}(g) = {<}[x]\}$

$f : P \longrightarrow Y$

There is a unique $F : X \longrightarrow Y$ s.t.

$\boxed{F(x) = f(F \restriction {<}[x])}$

This means we would need a more general version of the Recursion Theorem that does not require fixing the range in advance.

We need an extra axiom: the axiom scheme of Replacement.
(Skolem / Fraenkel 1922)

Jech.

1.7. Axiom Schema of Replacement. *If a class F is a function, then for any X there exists a set $Y = F(X) = \boxed{\{F(x) : x \in X\}}$.*

If φ is a formula in $n+2$ variables we say that φ is functional

$$F_\varphi \left[\begin{array}{l} \forall x \forall y \forall y' \forall p_1 \dots \forall p_n \\ \varphi(x, y, p_1, \dots, p_n) \wedge \varphi(x, y', p_1, \dots, p_n) \rightarrow y = y' \end{array} \right.$$

abbreviation

$$\mathrm{Repl}_\varphi \quad F_\varphi \longrightarrow \underline{\forall x} \forall p_1 \dots \forall p_n \underline{\exists r} \; \forall z \left(z \in r \longleftrightarrow \exists w \left(\underline{w \in x} \wedge \varphi(w, z, p_1, \dots, p_n) \right)\right)$$

Informally if φ behaves like a function, then for every x, if I restrict the fn def by φ to x, I get its range.

It's called Replacement for the following reason

$$x = \{\boxed{z} \,;\, z \in x\}$$

Replace z systematically by y ~~...~~ where $\varphi(z, y)$

[Jede: $F(z)$]

$$\{F(z) \,;\, z \in x\}$$

ZF_0 Zermelo-Fraenkel without Foundation

[Of course $ZF := ZF_0$ + Foundation, Lecture V].

Z + Repl.

ZF_0 proves a Recursion Theorem w/o fixed range.

Just to ease notation.

If φ is functional and $p_1, \dots, p_n$ are parameters, we write $\vec{p}$

$\varphi_x^{\vec{p}}$ for the unique y s.t.

$$\varphi(x, y, p_1, \dots, p_n)$$

<u>Recursion Theorem w/o fixed range</u> (ZF_0)

If φ is a functional formula and $(X, <)$ a wellorder and $p_1, \dots, p_n$ are parameters, then there is a unique function F with $\mathrm{dom}(F) = X$ and $\forall x \in X$

$$\boxed{F(x) = \varphi^{\vec{p}}_{F \restriction <[x]}}$$

RECURSION EQUATION.

[Follow the same proof strategy...]

Proof. As before, a GERM g is a function with $\mathrm{dom}(g) = <[x]$ for some $x \in X$ s.t. for all $z \in \mathrm{dom}(g)$

$$g(z) = \varphi^{\vec{p}}_{g \restriction <[z]}.$$

As before, we prove Lemmas 1 & 2 by induction.

L1 If g, g' are germs and $z \in \mathrm{dom}(g) \cap \mathrm{dom}(g')$, then $g(z) = g'(z)$.

L2 If $x \in X$, then there is a germ g s.t. $x \in \mathrm{dom}(g)$.

$$\Psi(x, y, \vec{p}) :\Longleftrightarrow \exists g \left(g \text{ germ} \wedge x \in \mathrm{dom}(g) \wedge y = g(x) \right)$$

We observe that Ψ is a functional formula. So apply Replacement to Ψ and get:

Replacement yields a set R s.t.

$$z \in R \longleftrightarrow \exists w \in X \; \Psi(w, z, \vec{p})$$

Now we can separate by Ψ from $X \times R$

$$F := \{(x,y) \in X \times R ; \Psi(x,y,\vec{p})\}.$$

q.e.d.

<u>Application</u>. Apply the Recursion Theorem to get the "next ordinal".

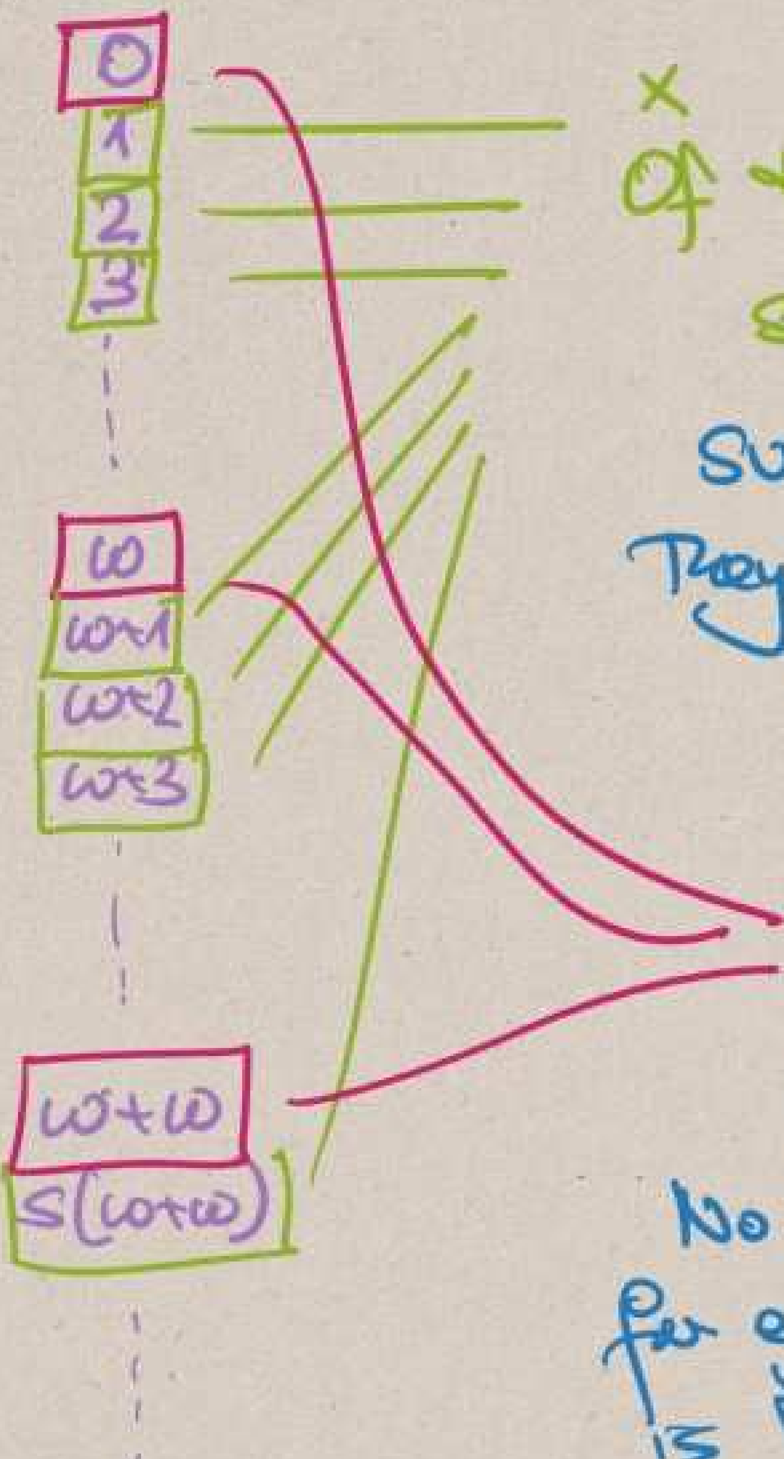
$$\varphi(x,y) :\Longleftrightarrow$$

$$(x = 0 \wedge y = \omega) \vee$$

$$\left(\begin{array}{c} x \text{ is a function with} \\ \mathrm{dom}(x) = n+1 \wedge \\ y = s(x(n)) \end{array} \right) \vee$$

$$(x \notin \mathbb{N} \wedge y = \emptyset)$$

$$\boxed{\begin{array}{l} g(0) := \omega \\ g(n+1) := s(g(n)) \end{array}}$$



x of the form $s(y)$ where $y \in x$

SUCCESSOR ORDINALS

They have a largest element.

not of the form $s(y)$.

LIMIT ORDINALS

No largest element: for every $\alpha \in \lambda$, there is $\beta \in \lambda$ with $\alpha \in \beta$

On ordinals, we often use a notational variant of induction and recursion, called TRANSFINITE INDUCTION

TRANSFINITE RECURSION

TRANSFINITE INDUCTION

Suppose α is an ordinal and $Z \subseteq \alpha$.

If $0 \in Z$ and

if $x \in Z$ then $s(x) \in Z$ and

if $x \in \alpha$ and x is a limit ordinal and $x \subseteq Z$, then $x \in Z$.

Then $Z = \alpha$.

TRANSFINITE RECURSION

α is an ord.

If x_0 is a set and for each x, we have some assigned f_x and for each function g, we have some assigned h_g

Then there is a unique F with $\mathrm{dom}(F) = \alpha$ s.t.

$$(*) \quad \begin{cases} F(0) := x_0 \\ F(s(x)) := f_{F(x)} \\ F(y) := h_{F \restriction y} & \text{if } y \text{ is a limit ord.} \end{cases}$$

Notational trick. Since the function F with $\mathrm{dom}(F) = \alpha$ and F' with $\mathrm{dom}(F') = \alpha'$ satisfying $(*)$ agree on the common domain by uniqueness, we can think of this as operation on all ordinals.

Applications

ORDINAL ADDITION

$$\alpha + 0 := \alpha$$
$$\alpha + s(\beta) := s(\alpha + \beta)$$
λ limit $\qquad \alpha + \lambda := \bigcup \{\underline{\alpha + \xi \,;\, \xi \in \lambda}\}$

$$1 + \omega = \bigcup\{\underline{1 + n \,;\, n \in \omega}\} = \mathbb{N} = \omega$$
$$\omega + 1 = \omega + s(0) = s(\omega + 0) = s(\omega) \neq \omega.$$

ORDINAL MULTIPLICATION

$$\alpha \cdot 0 := 0$$
$$\alpha \cdot s(\beta) := \alpha \cdot \beta + \alpha$$
λ limit $\qquad \alpha \cdot \lambda := \bigcup \{\alpha \cdot \xi \,;\, \xi \in \lambda\}$

ORDINAL EXPONENTIATION

$$\alpha^0 := 1$$
$$\alpha^{s(\beta)} := \alpha^\beta \cdot \alpha$$
λ limit $\qquad \alpha^\lambda := \bigcup \{\alpha^\xi \,;\, \xi \in \lambda\}$

$$\omega \cdot 2 = \omega \cdot s(1) = \omega \cdot 1 + \omega = \omega + \omega$$
$$\omega \cdot 1 = \omega \cdot s(0) = \omega \cdot 0 + \omega = 0 + \omega \overset{!}{=} \omega$$
$$2 \cdot \omega = \bigcup\{\underline{2 \cdot n \,;\, n \in \omega}\} = \mathbb{N} = \omega \neq \omega + \omega.$$