SET THEORY

MASTER MATH 2021/22

FIRST LECTURE 13 September 2021

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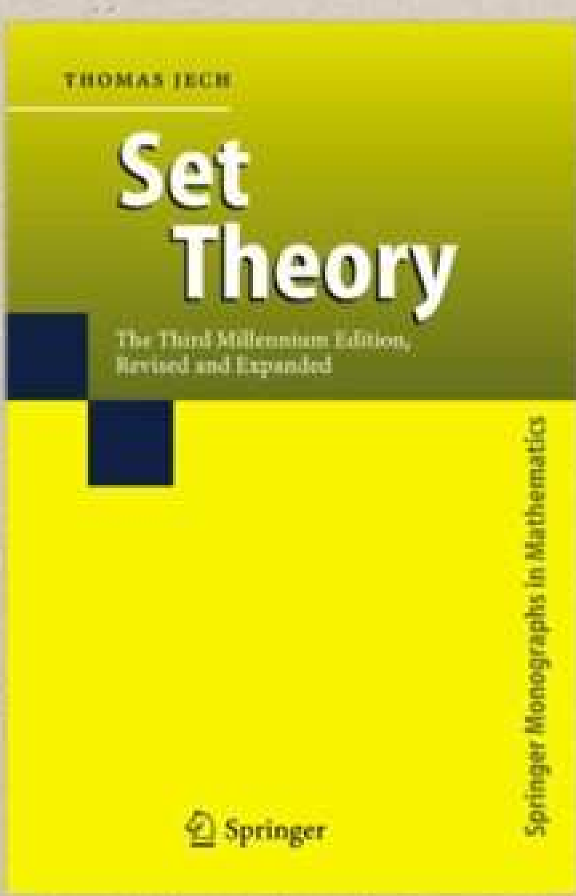
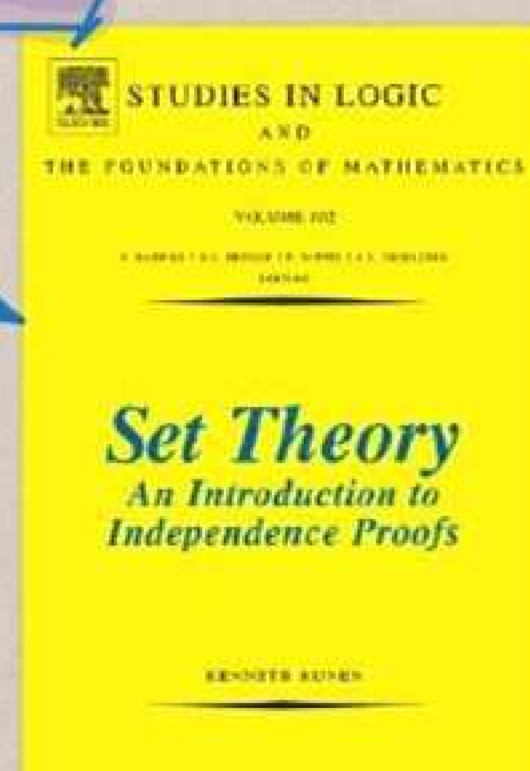
$\boxed{1-7}$ $\boxed{8-11}$ $\boxed{12-15}$

BL KP





EXAM 90%	COURSE WORK 10%

HOMEWORK
UNDERSTANDING
GROUP INTERACTION

Currently planned
<u>in person</u>

OPEN BOOK

- all materials allowed
- no help from other people allowed

GROUP INTER-ACTIONS:

elo page
→ selection of group interaction time slots

Homework: normal maths homework

Understanding:
once every two week

group interactions:
once every week

Marking of Coursework components:

Homework:
- not marked individually
- just count how many are done.

3-5 Q

$\sim 14 \times (3-5)$ → GOOD SOLUTIONS will be identified by the TA and posted (with permission of the author) on elo.

Understanding:
- fully marked
- according to the marking scheme of the exam.

full score: $3\frac{1}{2}$

satisfactory: 3

Group Interaction:
- participation

max. 10

RUDIMENTS OF AXIOMATIC SET THEORY

only UvA

Just the first 7 weeks

EXAM only about first 7 weeks

HW only H1–H7

Und. only U1–U3

GT multiply by 2, max. 10

Decision to do Rudiments exam:

~ end Nov / early Dec

EXAMS Set Thy 17 Jan / 21 Feb

Rudiments 20 Dec / 26 Mar

OVERVIEW OF THE FIRST 7-9 WEEKS

Part I. Basic Set Theory

First 7 Lectures

§1-4
first part of §5
§6

Lectures 8-11

second part of §5
maybe §8
§9

AXIOM SYSTEMS

(+AC) FST FINITE SET THEORY

(+AC) Z ZERMELO

(+AC) ZF Z.-FRAENKEL

FST

FST is strictly weaker than Z

Z

Z is strictly weaker than ZF

ZF

[ZF is strictly weaker than ZFC]

ZF+AC = ZFC

Famous open question, solved by Paul Cohen 1963.

Two PROTAGONISTS of Set Theory

ORDINAL NUMBERS
(counting beyond ∞)
"transfinite"

countable
uncountable
"$\mathbb{R}$ is uncountable"

CARDINAL NUMBERS
(different sizes of infinite sets)

THE AXIOMATIC METHOD

Euclid

(4th century BC)

ELEMENTS

AXIOMS "Ὅροι".

"A point is that of which there is no part."

POSTULATES

Αἰτήματα.

Αἰτήματα.

α'. Ἠιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

β'. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ' εὐθείας ἐκβαλεῖν.

γ'. Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.

δ'. Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

ε'. Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ' ἄπειρον συμπίπτειν, ἐφ' ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

Postulates

1. Let it have been postulated† to draw a straight-line from any point to any point.
2. And to produce a finite straight-line continuously in a straight-line.
3. And to draw a circle with any center and radius.
4. And that all right-angles are equal to one another.
5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).‡

† The Greek present perfect tense indicates a past action with present significance. Hence, the 3rd-person present perfect imperative Ἠιτήσθω could be translated as "let it be postulated", in the sense "let it stand as postulated", but not "let the postulate be now brought forward". The literal translation "let it have been postulated" sounds awkward in English, but more accurately captures the meaning of the Greek.

‡ This postulate effectively specifies that we are dealing with the geometry of *flat*, rather than curved, space.

David HILBERT (1862-1943)

Famous non-quote (?):

MAN MUSS JEDERZEIT ANSTELLE
VON

PUNKTEN → STÜHLE (chairs)

GERADEN → TISCHE (tables)

&

EBENEN → BIERSEDEL (beer mugs)

SAGEN KÖNNEN.

EXAMPLE FOR THE AXIOMATIC METHOD

AFFINE PLANE GEOMETRY

Axioms that try to capture $\mathbb{R}^2$ with its structure of points & lines.

Line ℓ is a subset of $\mathbb{R}^2$ defined by a linear equation, say

$$y = ax+b. \quad \ell = \{(x,y);\ ax+b=y\}$$

Famous open question in plane geometry:

Does the parallel postulate follow from the others?

<u>Technique</u> Axiomatise plane geometry by axioms A1, ..., A4 and parallel postulate A5 and develop a model that satisfies A1–A4 but not A5.

Def. Axioms of plane geometry

Chairs $\mathcal{C}$ Relation $I \subseteq \mathcal{C} \times \mathcal{T}$

Tables $\mathcal{T}$ expressing "lies on".

A1 $\forall C_0, C_1 \in \mathcal{C} \ \exists! T \in \mathcal{T}$
$C_0 I T$ and $C_1 I T$.

A2 $\forall T_0, T_1 \in \mathcal{T}$ exactly one of the following is true:

(a) $T_0 = T_1$

(b) $\exists! C \in \mathcal{C} \quad C I T_0$ and $C I T_1$

(c) $\forall C \in \mathcal{C}$ not ($C I T_0$ and $C I T_1$)

PARALLEL

A3 $\forall T \in \mathcal{T} \ \exists C_0, C_1 \in \mathcal{C}$ s.t.
$C_0 \neq C_1$ and $C_0 I T$ and $C_1 I T$

A4 $\exists C_0, C_1, C_2 \in \mathcal{C}$ s.t. $\forall T \in \mathcal{T}$
not ($C_0 I T$ and $C_1 I T$ and $C_2 I T$)

Structures satisfying A1–A4 are called AFFINE PLANES.

AS $\forall T \in \mathcal{T}\ \forall C \in \mathcal{E}$ s.t. not $C I T$
$\exists!\, T' \in \mathcal{T}$ s.t. T' is parallel to T and $C I T'$

Q. Does every affine plane satisfy AS?

A. NO!

Proof sketch Let U be the open unit disk: $U := \{(x,y);\ x^2+y^2<1\}$

$\mathcal{E} := U$

$\mathcal{T} := \{T;$ there is a line ℓ s.t. $T = \ell \cap U\}$

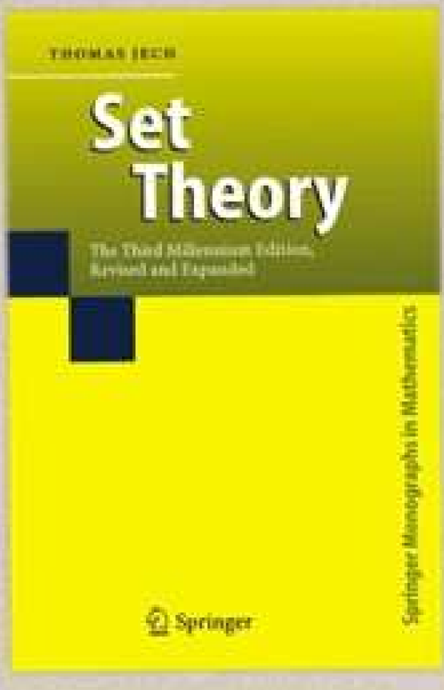
Three different tables all parallel (w/o chair in common) with table and containing chair.

Two modes of thought:

PRETHEORETICAL ← intuitions, desires, intended applications

AXIOMATISATION

MATHEMATICAL



Axioms of Zermelo-Fraenkel

1.1. *Axiom of Extensionality.* *If X and Y have the same elements, then $X = Y$.*

1.2. *Axiom of Pairing.* *For any a and b there exists a set $\{a, b\}$ that contains exactly a and b.*

1.3. *Axiom Schema of Separation.* *If P is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X : P(u, p)\}$ that contains all those $u \in X$ that have property P.*

1.4. *Axiom of Union.* *For any X there exists a set $Y = \bigcup X$, the union of all elements of X.*

1.5. *Axiom of Power Set.* *For any X there exists a set $Y = P(X)$, the set of all subsets of X.*

1.6. *Axiom of Infinity.* *There exists an infinite set.*

1.7. *Axiom Schema of Replacement.* *If a class F is a function, then for any X there exists a set $Y = F(X) = \{F(x) : x \in X\}$.*

1.8. *Axiom of Regularity.* *Every nonempty set has an $\in$-minimal element.*

1.9. *Axiom of Choice.* *Every family of nonempty sets has a choice function.*

The theory with axioms 1.1–1.8 is the Zermelo-Fraenkel axiomatic set theory ZF; ZFC denotes the theory ZF with the Axiom of Choice.

Reconstruct this by thinking about the axiomatic method.

1st phase: Pretheoretical:
what do we want to capture?

2nd phase: Axioms.

3rd phase: Study structures satisfying the axioms

Pre-theoretical mode:

set-theoretical vocabulary is ubiquitous in mathematics:

constant symbol? $\longrightarrow$ $\emptyset$

binary function symbol? $\longrightarrow$ $x \cup y$

$x \cap y$

function symbol? $\longrightarrow$ $\bigcup_{i \in I} x_i$

$\bigcap_{i \in I} x_i$

unary function symbol $\longrightarrow$ $\{x\}$

binary function sym. $\longrightarrow$ $\{x, y\}$

$\underline{\{x \,;\, P(x)\}}$ — ∞ many constant symbols, one for each property P.

typical set-theoretic notations

Our axioms should guarantee that objects reflecting these notations exist in our structures.

What is the first-order language?

Instead of using complicated languages with lots of symbols, we observe that all of concepts in our list can be expressed in a language with one binary relation symbol interpreted as

ELEMENT OF

We write $\in$ for this symbol

element symbol
epsilon symbol

$LST = \mathcal{L}_{\in}$ is the first order language with one binary rel. symbol $\in$.

What are the LST-structures?

They are $G = (V, E)$

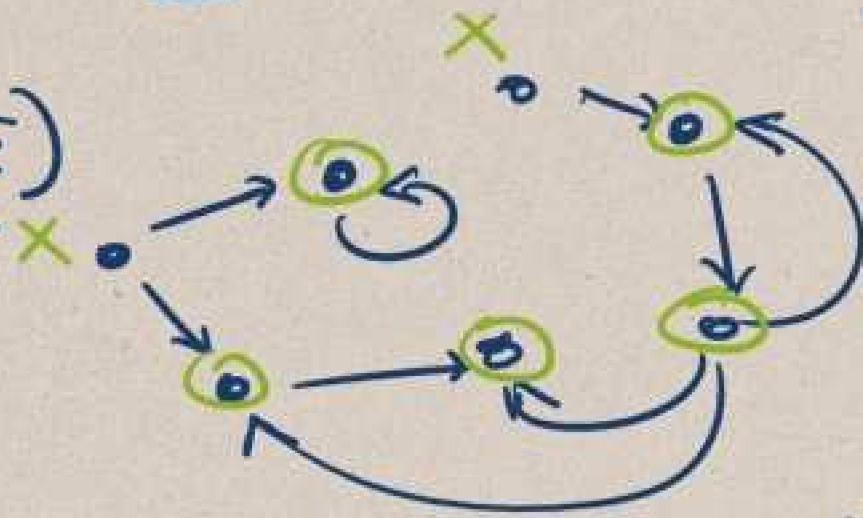
"universe"

binary relation

There are actually

DIRECTED GRAPH (DIGRAPHS)

$G = (V, E)$



This is a structure in the language of set theory where

$$v \longrightarrow w$$

is interpreted as

v is an element of w

First Axiom

SHORT-HAND

$z \notin e$

(Emp) $\exists e \, \forall z \, (\neg \, z \in e)$

The empty set axiom

"There is an empty set"

Second axiom

(Ext) $\forall x \forall y \left(x = y \longleftrightarrow \forall z \left(z \in x \leftrightarrow z \in y \right) \right)$

Extensionality

To summarise:

$\mathbb{G}_3$ from the last page is a model of (Emp) but not of (Ext).

<u>Corollary</u> The "set theory" {Emp, Ext} is strictly stronger than the "set theory" {Emp}.

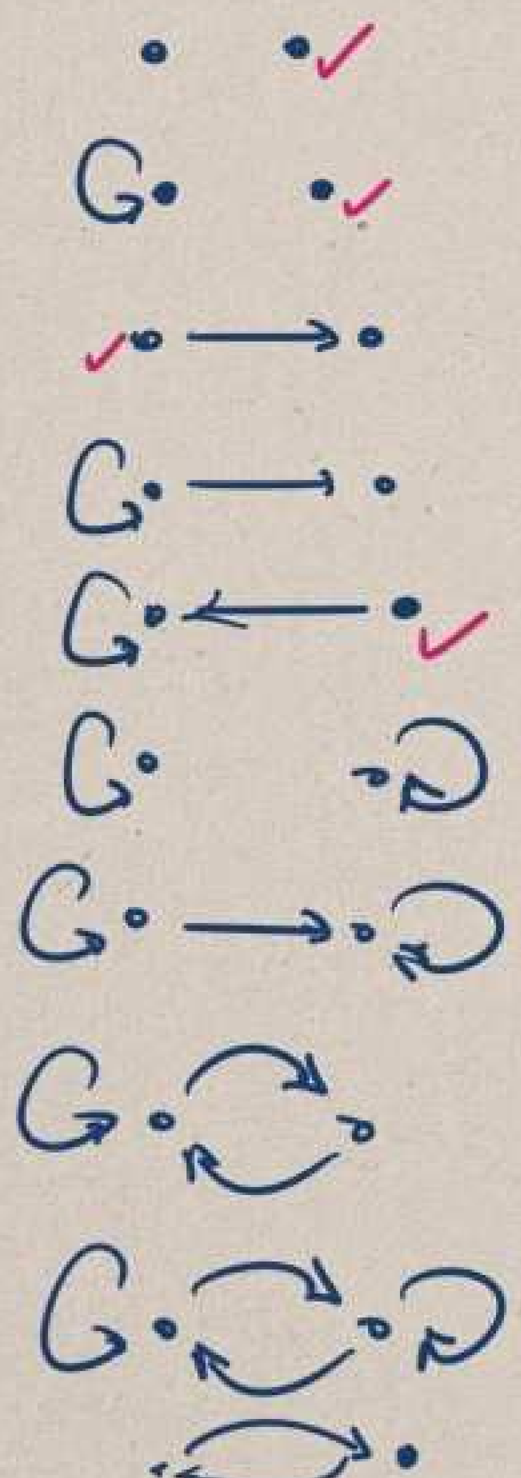
Warm-up exercise

All graphs with one & two points.

Emp Ext

~~Emp~~ Ext

one pt models imply $\{Ext\}$ is weaker than $\{Ext, Emp\}$



Emp ~~Ext~~

Emp Ext

Emp Ext

~~Emp~~ ~~Ext~~

Emp Ext

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~~Emp~~ Ext

~~Emp~~ Ext

~~Emp~~ ~~Ext~~

~~Emp~~ Ext

Other axioms

INTUITION $p = \{x, y\}$

(Pair) Axiom of pairing

$$\forall x \forall y \exists p \; \forall z \, (z \in p \leftrightarrow z = x \vee z = y)$$

(Union) Axiom of union

$$\forall x \exists u \; \forall z \, (z \in u \leftrightarrow \exists y \, (y \in x \wedge z \in y))$$

[HW set #1 also has "binary union"

$$\forall x \forall y \exists u \; \forall z \, (z \in u \leftrightarrow z \in x \vee z \in y)$$]

Once you have Ext + Emp / Pair / Union, you introduce $\emptyset$ for "the unique object satisfying Emp"

$\bigcup x$ for "the unique object satisfying Union".

$\{x, y\}$ for "the unique object satisfying Pair

Theorem If G is a digraph satisfying Ext, Pair, ~~Emp~~, then V is infinite.

$G = (V,E)$

[I.o.w.: No finite graph can be a model of all three axioms.]

Proof. We will prove that there is an injection from $\mathbb{N}$ into V.

Construct this by recursion.

Note that with (Pair) we also get singletons [apply Pair to $x=y$].

By Ext + Emp, we find a unique v in V that has no predecessor.

By Ext + Pair, we find for each $w \in V$ some $\overline{w} \in V$ s.t. $\overline{w}$ has exactly w as predecessor.

$f: \mathbb{N} \longrightarrow V$

$f(0) := v$

$f(n+1) := \overline{f(n)}$

Proof that f is an injection is by induction:

Prove that

$$\forall i\ \forall j<i\ \ f(j) \neq f(i)$$

by induction on i.

Let i_0 be the smallest counterex.
There is $j < i_0$ s.t.

$$f(j) = f(i_0)$$

Clearly $i_0 \neq 0$. But if $k>0$, then $f(k)$ is not the empty set, so $j>0$.

Let j' and i' be s.t.

$$j'+1 = j$$

$$i'+1 = i_0$$

By definition of $f(k+1)$ we get $j'=i'$

$\Longrightarrow$ $j = i_0$. Contradiction!

q.e.d.

Last axiom candidate

Pretheoretically, we want $\{x ; P(x)\}$

where P is a property

$\updownarrow$

formula

If φ is a formula, then

$$(\mathrm{Comp}_\varphi) \quad \exists c \forall z \, (z \in c \longleftrightarrow \varphi(z))$$

COMPREHENSION FOR φ

If $\mathfrak{S} \models \mathrm{Ext} + \mathrm{Comp}_\varphi$, then we are allowed to use

$$\boxed{\{x ; \varphi(x)\}}$$

Do we actually mean this in ordinary mathematics when we write $\{(x,y) ; x^2+y^2 < 1\}$.

No, we mean $\{(x,y) \in \mathbb{R}^2 ; x^2+y^2 < 1\}$

Theorem (Russell, 1901)

There is a φ s.t. no graph $\mathbb{G}$ satisfies (Compφ).

Proof. Let φ be the "Russell formula":

$$z \notin z.$$

Then if $\mathbb{G} \models \text{Comp}\varphi$, there is a vertex $r \in V$ s.t. r satisfies (Compφ):

$$\forall z \, (z \in r \longleftrightarrow z \notin z)$$

Plug in r for z:

$$r \in r \longleftrightarrow r \notin r.$$

Contradiction!

q.e.d.

SOLUTION TO THIS PROBLEM

(maxim "one step back from disaster" in philosophy of maths)

The Axiom Scheme of Separation

Fix a formula φ with $n+1$ free variables

(Sep φ)

$$\boxed{\forall x \, \forall p_1 \cdots \forall p_n \, \exists s \, \forall z \, (z \in s \longleftrightarrow z \in x \wedge \varphi(z, p_1, \ldots, p_n))}$$

Check that the contradiction in Russell's Theorem doesn't work anymore.

$$\varphi := z \notin z$$

Let $x \in V$. Then (Sep φ) gives me $s \in V$ s.t.

$$\forall z \, (z \in s \longleftrightarrow z \in x \wedge z \notin z)$$

Set $z := s$

$$s \in s \longleftrightarrow s \in x \wedge s \notin s$$

NOT A CONTRADICTION.

Next time

Definition If $G=(V,E)$ is a graph, we call $v \in V$ a *universal vertex* if for all $w \in V$, $w E v$.

Theorem If G satisfies the Axiom Scheme of Separation, then G has no universal vertex.

Remark Often expressed as "there is no set of all sets".

For homework:

Note that Separation implies Emp.
If φ is the formula

$$z \neq z$$

then separating from any set by φ gives you set without elements.