Comments on Understanding & Explanation Tasks

MasterMath: Set Theory
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Understanding & Explanation tasks ask you to describe either mathematical concepts or mathematical proofs in your own words. These tasks correspond to the exam questions in **Part I** of the exam which is worth 7 out of 10 points and will decide whether you pass or fail the exam component. **Part I** of the exam will have two questions, each worth $3\frac{1}{2}$ points, therefore we also give $3\frac{1}{2}$ points for each of the **U** tasks. Your answer will be marked according to whether it is **correct**, **comprehensive**, and **well-structured**. An answer is *comprehensive* if all of the important mathematical ideas are discussed and explained. In the exam, if both of your answers in Part I are marked as **satisfactory**, you are guaranteed to get a passing exam mark.

An answer will be considered **good** if all three criteria are satisfied. These answers will get full points (i.e., $\mathbf{3\frac{1}{2}}$ **points**).

It will be considered **satisfactory** if it has minor deficiencies in some of the three criteria. E.g., fixable errors in definitions or arguments would be considered a minor deficiency in correctness, the omission of one among several ideas or proof steps would be considered a deficiency in comprehensivity, a general lack of structure or confused prose would be considered a deficiency in being well-structured. Satisfactory answers will get **3 points**.

It will be considered **unsatisfactory** if it has a major deficiency in either correctness or comprehensivity, e.g., a flaw in a definition that invalidates the argument, a major error in an argument, or omitting the main idea of the proof would be considered major deficiencies. Unsatisfactory answers will get either **2 points**, **1 point**, or **0 points**, depending on the flaws.

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(1) *Understanding & Explanation tasks* are not about knowledge or speed. Since having a **well-structured** answer is part of the criteria, your first step should be to *structure* what you want to say.

(2) In the case of **U1**, first identify the subtasks listed on the sheet:

  (i) Explain how one can show that very weak axiom systems of set theory cannot prove σ.
  (ii) Also explain which axioms are sufficient to prove σ.
  (iii) Give concrete examples of axioms systems with these properties as well as proof sketches of these facts.

  So, there are two explanation tasks and a request to give examples. Decide how you wish to structure your answer and which examples to use. There are many ways to do this: let's give a few examples.

Example 1. You can have a brief explanation how to prove unprovability and provability and then give examples of axiom systems T and T' such that $T \vdash \sigma$ and $T' \nvdash \sigma$.

Example 2. You could structure your answer into two sections: unprovability and provability each with examples and proof sketches.

Example 3. You can follow the (i), (ii), (iii) structure of the task description, i.e., first given an explanation how to prove unprovability, then state which axioms prove σ and finally, give an example for (i) and proof sketches of the claims.

Note that the options will all provide the same mathematical content, but organise it in slightly different ways. There is no *correct* answer which one to pick: the most important task here is to think about the options and make a conscious decision (possibly with a reason) to pick one of them.

A well-structured answer does not jump back and forth between things that belong to different parts of the structure.

(3) Now that you have decided on the structure of your answer, you need to think about the mathematical content of your answer. In the case of **U1**, this is mainly the choice of the examples, i.e., an axiom system that proves σ and one that does not.

(4) At this point, you still have not written a word, but you know what you wish to say. The final step is to fill in the demands of the "proof sketches [with references] to the lecture notes". Open the lecture notes and find the right references for the claims you plan to use. (For **U1**, the lecture notes were fully sufficient, but in general, it might be useful to also consider the homework sheets, the group interaction sheets, or the book(s).)

(5) Now, you are prepared to write down your answer. Follow the structure that you decided on in step (2), fill in the mathematical content from (3), and, whenever you make a claim, refer to the lecture notes (or homework or group interaction or other literature).

On the following pages, you will find a good solution to **U1**: pages 3 and 4 (up to the horizontal line) give a minimal good solution for **U1** using the structuring option given in *Example 1* in (2) above. This solution is very terse and your solutions will probably contain more words and some more details. On page 4, you find some additional details for **2.** that you could have added and on page 5, you find some additional details for **3.** None of these details are necessary for the solution to be counted as "good".

Page 6 contains a correction of a claim on page 8 of the lecture notes of *Lecture II* that is somewhat related to the questions discussed here.

U1 MINIMAL SOLUTION

1. Proving unprovability.
If T is an axiom system and φ is any sentence, we prove that $T \nvdash \varphi$ by providing a model $\mathfrak{M} \models T$ such that $\mathfrak{M} \models \neg\varphi$.

2. Provability of σ.
In Lecture II (Lecture Notes II, page 12), it was proved that

$$\mathrm{FST} \vdash \sigma.$$

[In fact, σ is the special case $n = 1$ of the claim on p. 12.]

3. Unprovability of σ.
$T := \{\mathrm{Ext}, \mathrm{Pow}\}$
Then $T \nvdash \sigma$. By 1., we provide $\mathfrak{M} \models T \cup \{\neg\sigma\}$
In Lecture II (notes p. 7), we saw that

is a model with a one-element set that has exactly one subset.

But  satisfies

Pow (Lecture II, p. 12)
Ext (Lecture I, p. 18).

Possible additional detail (not needed):

Additional detail for 2.

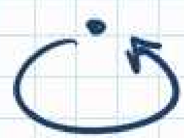
- In general, Ext gives an upper bound of 2^n for the size of a power set of a set with n elements. (Lecture II, p. 8, Remark 1)
- In general, Sep gives a lower bound of 2^n for the size of a power set of a set with n elements. (Lecture II, p. 8, Remark 2)
- Therefore, $T := \{\text{Ext}, \text{Sep}\}$ proves σ (and the generalisation of σ for n-element sets).
- Instead of Sep, we can use Pair, Union, and Emp. (Lecture II, p. 8, Remark 2).

[Cf. remark on p. 4.]

Additional detail for 3.

The graph  satisfies also Pair, Union and Sep, so these could also be listed as part of the example, but since this was not mentioned explicitly in the lectures, it's not so easy to refer to it.

One way to do so would be:

It is easily checked that  satisfies Pair, Union, and Sep.

But if say so, be sure that the claim <u>is</u> correct and easily checked.

Remark on lower bounds for the size of $P(x)$:

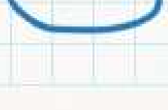
Remark 2 on p. 8 of Lecture II claims that we can use Pair + Union instead of Sep to get a lower bound of 2^n.

As discussed after Lecture III, this is not quite correct: by repeated use of Pair + Union, e.g.,

$$
\begin{aligned}
x, y &\longmapsto \{x, y\} =: a \\
y, z &\longmapsto \{y, z\} =: b \\
a, b &\longmapsto \{a, b\} = \{\{x, y\}, \{y, z\}\} =: w \\
w &\longmapsto \bigcup w = \{x, y, z\}
\end{aligned}
$$

we can get all NON-EMPTY finite subsets of a set, so we get a lower bound of $\boxed{2^n - 1}$

But to get the empty set, we need either Emp or Sep.

And our model  is an example of this!