Understanding & Explanation Task **U5**

MasterMath: Set Theory
2021/22: 1st Semester
K. P. Hart, Steef Hegeman, Benedikt Löwe & Robert Paßmann

Deadline for Understanding & Explanation Task U5: Monday, 22 November 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

Understanding & Explanation tasks (U).

Marking Scheme.

An answer will be considered **good** if all three criteria are satisfied. These answers will get full points (i.e., $\mathbf{3\frac{1}{2}}$ **points**).

It will be considered **satisfactory** if it has minor deficiencies in some of the three criteria. Satisfactory answers will get **3 points**.

It will be considered unsatisfactory if it has a major deficiency in either correctness or comprehensivity. Unsatisfactory answers will get either **2 points**, **1 point**, or **0 points**, depending on the flaws.

Task U5: In class we discussed closed unbounded sets and stationary sets for regular uncountable cardinals. One of the main results is that if κ is regular and uncountable then the intersection of fewer than κ many closed and unbounded sets is again closed and unbounded.

Explain where both properties are used in the proof and give counterexamples for the cases where one of the assumptions 'uncountable' or 'regular' is dropped.

Solution

The proof that the intersection of two closed unbounded sets is again closed and unbounded used an increasing ω-sequence of ordinals and the fact that the supremum of this sequence was smaller than κ. This requires that the cofinality of κ is uncountable.
The proof for a collection of λ closed unbounded sets, where $\lambda < \kappa$, built an increasing sequence of ordinal length $\lambda \times \omega$ (ω copies of λ). The proof used ω many times that the supremum of a sequence of length λ is smaller than κ and in the end that the supremum of a countable sequence is smaller than κ as well. This requires that the cofinality of κ is equal to κ and uncountable.

The cardinal/ordinal ω_0 has disjoint closed and unbounded sets, namely the sets E and O of even and odd natural numbers respectively.

In case κ is not regular we take an increasing cofinal sequence $\langle \gamma_\alpha : \alpha < \operatorname{cf} \kappa \rangle$ in κ. The final segments $[\gamma_\alpha, \kappa)$ are closed and unbounded and the intersection $\bigcap_{\alpha < \operatorname{cf} \kappa} [\gamma_\alpha, \kappa)$ is empty.