Understanding & Explanation Task U4

MasterMath: Set Theory
2021/22: 1st Semester
K. P. Hart, Steef Hegeman, Benedikt Löwe & Robert Paßmann

Deadline for Understanding & Explanation Task U4: Monday, 8 November 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

Understanding & Explanation tasks (U).

Marking Scheme.

An answer will be considered **good** if all three criteria are satisfied. These answers will get full points (i.e., **$3\frac{1}{2}$ points**).

It will be considered **satisfactory** if it has minor deficiencies in some of the three criteria. Satisfactory answers will get **3 points**.

It will be considered unsatisfactory if it has a major deficiency in either correctness or comprehensivity. Unsatisfactory answers will get either **2 points**, **1 point**, or **0 points**, depending on the flaws.

Task U4: Explain how the evaluation of $2^{\aleph_\omega}$ proceeds in all four combinations of the following circumstances

- $2^{\aleph_0} < \aleph_\omega$ versus $2^{\aleph_0} > \aleph_\omega$
- in general versus if SCH holds

Also explain why is there not a third possibility in the first item?

Solution

To answer the last question first: for all κ the cofinality of 2^κ was shown to be larger than κ. So, in particular, the cofinality of $2^{\aleph_0}$ is uncountable and hence $2^{\aleph_0}$ is not equal to $\aleph_\omega$.

We distinguish the cases 'in general' and 'SCH holds' separately.

In general. In this case the distinction $2^{\aleph_0} < \aleph_\omega$ versus $2^{\aleph_0} > \aleph_\omega$ is a red herring.
The evaluation of $2^{\aleph_\omega}$ uses the limit behaviour of the sequence $\langle 2^{\aleph_n} : n \in \omega\rangle$.
If it is constant from some m not, say $2^{\aleph_n} = \kappa$ for $n \geqslant m$ then $2^{\aleph_\omega} = \kappa$.
If it is not constant from some m on then $2^{\aleph_\omega} = \gimel(\lambda)$, where $\lambda = \sup_{n\in\omega} 2^{\aleph_n}$.
But $\operatorname{cf}\lambda = \aleph_0$, hence $2^{\aleph_\omega} = \lambda^{\aleph_0}$.

If SCH holds. In this case the answer is the same if $\langle 2^{\aleph_n} : n \in \omega\rangle$ is constant from some m on.
In the other case we have $2^{\aleph_\omega} = \lambda^+$, by the SCH. This because $\operatorname{cf}\lambda = \aleph_0$ and so $\lambda^{\aleph_0} = \lambda^+$.
Again, we see that the value of $2^{\aleph_0}$ has no real bearing on the outcome, except that in case $\lambda = \aleph_\omega$ we find $2^{\aleph_\omega} = \aleph_{\omega+1}$.