Homework 9 [redacted]

Exercise 1

a) suppose $p \in I$ and $q \in J$

proof: then A_p contains a cub set

B_q contains a cub set

Since the intersection of two cub sets is again a cub set, which is not empty,

$A_p \cap B_q \neq \emptyset$

There exists $\alpha \in A_p \cap B_q$. Then $f(\alpha) < p$ and $f(\alpha) > q$

Thus $p > q$.

Therefore, if $p \in I$ and $q \in J$, then $p > q$.

b) suppose $I = \mathbb{Q}$

proof: then $\bigcap_{q \in I} A_q = \bigcap_{q \in \mathbb{Q}} A_q \neq \emptyset$ as the intersection of countably many cub sets of ω_1 is still cub.

There exist $\alpha \in \bigcap_{q \in \mathbb{Q}} A_q$

then $f(\alpha) < q$ for any $q \in \mathbb{Q}$ which is impossible.

Thus $I \neq \mathbb{Q}$

Similarly, we can prove that $J \neq \mathbb{Q}$

c) The case when $J = \emptyset$ or $I = \emptyset$ is easy as $I, J \neq \mathbb{Q}$, I is upward closed and J is downward closed

proof: suppose $J \neq \emptyset$ and $I \neq \emptyset$

By a), for any $p \in I$, $q \in J$, $p > q$.

Thus $\sup J \leq \operatorname{Inf} I$.

Suppose $\sup J = \operatorname{Inf} I = a \in \mathbb{R}$

Then as the intersection of countably many cub sets of ω_1 is still cub,

$\bigcap_{p \in I} A_p \cap \bigcap_{q \in J} B_q$ contains a cub (of cardinality $\aleph_1$)

For any $x \in \bigcap_{p \in I} A_p \cap \bigcap_{q \in J} B_q$, $p > f(x) > q$ for any $p \in I$, $q \in J$

Thus $\operatorname{Inf} I \geq f(x) \geq \sup J$

$f(x) = a$

This contradicts the assumption that f is injective.

Therefore, $\sup J < \operatorname{Inf} I$

d)

proof: By c), $\sup J < \operatorname{Inf} I$

Thus there exists $q \in \mathbb{Q}$ s.t $\sup J < q < \operatorname{Inf} I$ as $\mathbb{Q}$ is dense in $\mathbb{R}$.

Then there exist $a, b \in \mathbb{Q}$ s.t $\sup J < a < q < b < \operatorname{Inf} I$

As $a \notin J$, B_a doesn't contain any cub set, for any cub set C, there exists $x \in C$ s.t $f(x) \leq a < q$, namely $x \in A_q$

As $b\notin I$, A_b doesn't contain any cub set, for any cub set C, there exists $x\in C$ s.t $f(x)\geq b>q$, namely $x\in B_q$.
Thus A_q and B_q are stationary.

Exercise 2

proof: we prove by induction on α that for every stationary subset S of ω_1, there is a closed subset of order type $\alpha+1$ that is contained in S.
$\alpha=0$, clearly true (a single element of S works)
$\alpha=\beta+1$, by IH, there's a closed subset Γ of order type $\beta+1$ that is contained in S.
As $\alpha<\omega_1$, Γ is countable and thus $\sup\Gamma<\omega_1$
Then $[\sup\Gamma+1,\omega_1)$ is a cub set.
As S is stationary, $S\cap[\sup\Gamma+1,\omega_1)\cap\neq\emptyset$
Let x be the least element s.t $x\in S\cap[\sup\Gamma+1,\omega_1)$, then $\Gamma\cup\{x\}$ is of order type $(\beta+1)+1$ that is contained in S. As Γ is closed, and $x>\sup\Gamma$, $\Gamma\cup\{x\}$ is closed as well
$\alpha=\xi$ is a limit ordinal, as $\alpha<\omega_1$, $cf(\alpha)=\omega$
Let $\langle\alpha_n:n\in\omega\rangle$ be an increasing and cofinal sequence in α.
Define a sequence $\langle C_\gamma:\gamma<\omega_1\rangle$ of countable closed subsets stated in the hint recursively as follows:
Suppose $\langle C_\xi:\xi<\gamma\rangle$ has been defined s.t $C_\xi\subseteq S$ for any $\xi<\gamma$, and $\max C_{\delta_1}<\min C_{\delta_2}$ whenever $\delta_1<\delta_2$ and if $\gamma'=\omega\cdot\delta+n$, then $C_{\gamma'}$ has order type α_n+1. Consider γ, by division algorithm, there exists unique δ and n s.t $\gamma=\omega\cdot\delta+n$ $(n<\omega)$
Since $\delta<\omega_1$ and C_ξ is countable for any $\xi<\gamma$, $m=\bigcup_{\xi<\gamma}C_\xi$ is countable $<\omega_1$
$[m+1,\omega_1)$ is closed and unbounded in ω_1
As S is stationary, $S\cap[m+1,\omega_1)$ is stationary as well.
As $[m+1,\omega_1)$ is of order type ω_1, $S\cap[m+1,\omega_1)$ is stationary, by IH there exists a closed subset $C_\gamma\subseteq[m+1,\omega_1)\cap S$ of order type α_n+1. Clearly for any $\xi<\gamma$, $\max C_\xi<\min C_\gamma$. C_γ satisfies the properties we want.
Now let $\Sigma=\{x\mid x$ is the limit point of $\{\max C_\gamma:\gamma<\omega_1\}\}$
Σ is unbounded: for any $\delta<\omega_1$, $\max C_\delta\geq\delta$ as $\delta\mapsto\max C_\delta$ is order-preserving.
Suppose $\delta=\omega\cdot\beta+n$, then $\bigcup_{n<\omega}\max C_{\omega\cdot\beta+n}$ is a limit point and thus belongs to Σ
$\bigcup_{n<\omega}\max C_{\omega\cdot\beta+n}>\delta$.

Since the limit point of limit points of $\{\max C_\gamma:\gamma<\omega_1\}$ is clearly the limit point of $\{\max C_\gamma:\gamma<\omega_1\}$, Σ is closed and thus a cub set.
As S is stationary, $k\in S\cap\Sigma\neq\emptyset$
Now as k is a limit point of $\{\max C_\gamma:\gamma<\omega_1\}$, $k=\sup_{n<\omega}\max C_{\gamma_n}$ as $cf\,k=\omega$

where C_{γ_n} is of order type α_n+1.

Now take C_{γ_0} whose order type is α_0+1, take the first ξ_1 many elements from C_{γ_1} s.t $\alpha_0+1+\xi_1=\alpha_1+1$ (ξ_1 is a successor ordinal and such ξ_1 exists and is unique by ordinal subtraction), then take the first ξ_2 many elements from C_{γ_2} s.t $\alpha_1+1+\xi_2=\alpha_2+1$ and so on.

Let T be the set of all elements taken from the above process.

$T\cup\{\kappa\}$ is order type $\alpha+1$

Since by our construction, for $\delta\neq\delta'$, C_δ and $C_{\delta'}$ are disjoint closed sets, and ξ_i's are all successor ordinals, T is a closed subset

As κ is the limit point of T, $T\cup\{\kappa\}$ is closed as well.

Clearly $T\cup\{\kappa\}\subseteq S$. Thus we get a closed subset of order type $\alpha+1$ contained in S and finishes the induction.

Exercise 3

a)

proof: Define $f: E^{\omega_2}_{\omega_1}\to\omega_2$ as follows:

$f(\alpha)=\sup(\alpha\cap C_\alpha)$

As C_α is countable and $cf\alpha=\omega_1$, $\sup(\alpha\cap C_\alpha)<\alpha$.

Thus f is regressive on $E^{\omega_2}_{\omega_1}$ which is stationary.

By the Pressing-down lemma, there exists $\gamma\in\omega_2$ s.t $T=\{\alpha \mid f(\alpha)=\gamma\}$ is stationary.

Now for any $\alpha\in T$, $\alpha\cap C_\alpha\subseteq\gamma+1$

$[\gamma+1,\omega_2)$ is a cub set, and thus $T\cap[\gamma+1,\omega_2)$ is stationary as well.

For any C_α, let $D_\alpha=[\sup C_\alpha+1,\omega_2)$ which is a cub set ($\sup C_\alpha<\omega_2$ as C_α is countable).

Consider $\Delta_{\alpha<\omega_2} D_\alpha=\{\xi \mid \forall\beta<\xi\to\xi\in D_\beta\}$, the diagonal intersection of cub sets is again a cub set.

Thus $\Sigma=T\cap[\gamma+1,\omega_2)\cap\Delta_{\alpha<\omega_2} D_\alpha$ is stationary, and thus is of cardinality $\aleph_2$

For any $\alpha,\beta\in\Sigma$ s.t $\alpha\neq\beta$

$\alpha\geq\gamma+1$, $\beta\geq\gamma+1$

W.l.o.G assume $\alpha<\beta$, as $\beta\cap C_\beta\subseteq\gamma+1$

$\alpha\notin C_\beta$.

Since $\beta\in\Delta_{\alpha<\omega_2} D_\alpha$, $\beta\in D_\alpha$ and thus $\beta\notin C_\alpha$.

Therefore, for any $\alpha,\beta\in\Sigma$ s.t $\alpha\neq\beta$, $\alpha\notin C_\beta$.

Σ is what we want.

b)

proof: If we only require $|S(\alpha)|\leq\lambda$, then consider the following counterexample:

$S(\alpha)=\alpha$ where $k=\omega_2$, $\lambda=\omega_1$ $|S(\alpha)|\leq\aleph_1$

It's clear that we can't find a free set for s of cardinality $\aleph_2$ in this case.

c)

proof: The proof is the same as a)

Let $\langle S_\alpha : \alpha<k\rangle$, define $f: E^k_\lambda \to k$ as follows: $f(\alpha)=\sup(\alpha\cap S_\alpha)$

As $|S_\alpha|<\lambda$ and $cf(\alpha)=\lambda$, $\sup(\alpha\cap S_\alpha)<\alpha$

Thus f is regressive on E^k_λ which is stationary.

There exists $\delta\in k$ s.t $T=\{\alpha| f(\alpha)=\delta\}$ is stationary by the Pressing down lemma.

For any $\alpha\in T$, $\alpha\cap S_\alpha\subseteq\gamma+1$

$[\gamma+1,k)$ is a cub set and thus $[\gamma+1,k)\cap T$ is stationary.

~~There exists $\delta\in k$ s.t $T=$~~ For any S_α, let $D_\alpha=[\sup S_\alpha+1,k)$ ($\sup S_\alpha<k$ as $k>\lambda>|S_\alpha|$ and k is regular)

D_α's are cub sets, so is $\underset{\alpha<k}{\Delta} D_\alpha$

Then $\Sigma=T\cap[\gamma+1,k)\cap\underset{\alpha<k}{\Delta} D_\alpha$ is stationary, and is thus of cardinality k.

It's easy to check that Σ is what we want.

d)

proof: If λ is finite, then $S: k\to[k]^{\leq n}$, $E^k_{\omega_0}$ will work as in a) and c)

If λ is infinite, for any $\alpha\in k$, $|S(\alpha)|<\lambda$ and thus $|S(\alpha)|^+<\lambda$

suppose for any regular cardinal μ below k, $|\{\alpha| |S(\alpha)|<\mu\}|<k$

Then $k=\left|\bigcup_{\substack{\mu<\lambda\\ \mu \text{ is regular}}}\{\alpha| |S(\alpha)|<\mu\}\right|$

$=\sum_{\substack{\mu<\lambda\\ \mu \text{ is regular}}}|\{\alpha| |S(\alpha)|<\mu\}|$ which is impossible as k is regular.

Thus there exists a regular cardinal $\mu<\lambda$ s.t $\{\alpha| |S(\alpha)|<\mu\}$ has cardinality k.

Let $f: k\to\{\alpha| |S(\alpha)|<\mu\}$ be a bijection.

Define $S': k\to[k]^{<\mu}$ as follows:

$S'(\alpha)=f^{-1}[S(f(\alpha))]$

Then by c), there exists a free set F for S' of cardinality k.

For any $\alpha,\beta\in F$, $\alpha\notin S'(\beta)$ iff $f(\alpha)\notin S(f(\beta))$

Thus $\{f(\alpha)|\alpha\in F\}$ is a free set for S of cardinality k.

e)

proof: Induction on λ

$\lambda=1$, trivially true (as every element is mapped to the empty set)

$\lambda=n+1$, suppose there exist m s.t for any $n>m$, $S(n)\cap m\neq\emptyset$.

Then define $S': \aleph_0 \to [\aleph_0]^{<n}$

$$S'(a) = \begin{cases} S(a)\restriction m & \text{if } a>m \\ \emptyset & \text{otherwise} \end{cases}$$

By IH, there exists F s.t for any $a,b \in F$ s.t $a \neq b$, $a \notin S'(b)$ and F is of cardinality $\aleph_0$.

Let $F' = F \cap [m+1, \omega)$ which is of cardinality $\aleph_0$

For any $a,b \in F'$ s.t $a \neq b$, $a>m$, $b>m$ $a \notin S'(b) = S(b)\restriction m$

Thus $a \notin S(b)$

Therefore, F' is a free set for S of cardinality $\aleph_0$.

suppose for any m, there exists n s.t $n>m$ and $S(n) \cap m = \emptyset$, then we defines $F: \omega \to \omega$ ~~left~~ recursively as follows:

$F(0) = 0$

$F(n+1) =$ the least element m s.t $m > \max(\bigcup_{i \in \mathrm{Ran} F\restriction_{<n+1}} S(i) \cup \mathrm{Ran} F\restriction_{<n+1}) + 1$ and $S(m) \cap \max(\bigcup_{i \in \mathrm{Ran} F\restriction_{<n+1}} S(i) \cup \mathrm{Ran} F\restriction_{<n+1}) + 1 = \emptyset$.

Then clearly F is injective and $|\mathrm{Ran}(F)| = \aleph_0$.

It's then easy to see that $\mathrm{Ran}(F)$ is a free set for S.

~~This fin~~ In either case, we get a free set for S of cardinality $\aleph_0$.

This finishes the induction and thus the whole proof.

Scanned with CamScanner