Homework Sheet #8

MasterMath: Set Theory
2021/22: 1st Semester
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Deadline for Homework Set #8: Monday, 8 November 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(27) Let κ be infinite. Prove that every ordinal $\alpha < \kappa^+$ can be written as the union of countably many sets, $\alpha = \bigcup_{n<\omega} X_{\alpha,n}$, such that for every n the order type of $X_{\alpha,n}$ is at most κ^n (ordinal power). *Hint*: This is easy if $\alpha \leqslant \kappa$; use induction above κ. Going from α to $\alpha+1$ shift the sets $X_{\alpha,n}$ one up and put α in $X_{\alpha+1,0}$; if α is a limit combine the earlier $X_{\beta,k}$ into sets $X_{\alpha,n}$ (and use that $\operatorname{cf}\alpha \leqslant \kappa$).

(28) Let λ be an infinite cardinal, $\langle \kappa_i : i < \lambda \rangle$ an increasing sequence of regular cardinals, and $\kappa = \sup_{i<\lambda} \kappa_i$. Give a direct proof of the formula

$$2^\kappa = \prod_{i<\lambda} 2^{\kappa_i}$$

by constructing a bijection between $\mathcal{P}(\kappa)$ and $\prod_{i<\lambda} \mathcal{P}(\kappa_i)$.

(29) Prove the following statements
a. $\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}$.
b. If $2^{\aleph_1} = \aleph_2$ and $\aleph_\omega^{\aleph_0} > \aleph_{\omega_1}$ then $\aleph_{\omega_1}^{\aleph_1} = \aleph_\omega^{\aleph_0}$.
c. If $2^{\aleph_0} \geqslant \aleph_{\omega_1}$ then $\gimel(\aleph_\omega) = 2^{\aleph_0}$ and $\gimel(\aleph_{\omega_1}) = 2^{\aleph_1}$.

(30) Prove: if β is such that $2^{\aleph_\alpha} = \aleph_{\alpha+\beta}$ for all α then $\beta < \omega$. Complete the following steps. Assume $\beta \geqslant \omega$.
a. Let α be minimal such that $\alpha + \beta > \beta$. Show that α is a limit.
b. Let $\kappa = \aleph_{\alpha+\alpha}$; show κ is singular.
c. Prove: $2^{\aleph_{\alpha+\xi}} = \aleph_{\alpha+\beta}$ whenever $\xi < \alpha$.
d. Calculate 2^κ and derive a contradiction.

Remark. It is consistent to have $2^{\aleph_\alpha} = \aleph_{\alpha+2}$ for *all* α.