Set Theory, Homework 8

[redacted]

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Exercise 27

Proposition. *Let κ be infinite. Then every ordinal $\alpha < \kappa^+$ can be written as the union of countable sets, $\alpha = \bigcup_{n<\omega} X_{\alpha,n}$; such that for every n the order type of $X_{\alpha,n}$ is at most κ^n (ordinal power).*

Proof. We can distinguish two cases:

- $\alpha \leq \kappa$: Then let $X_{\alpha,0} = 0$ and $X_{\alpha,n} = \alpha$ for $0 < n < \omega$. Then clearly $\alpha = \bigcup_{n<\omega} X_{\alpha,n}$ and for every $n > 0$ we have that $\alpha \leq \kappa^n$ as $\alpha \leq \kappa$. Furthermore, for $n = 0$ we see that $0 \leq \kappa^0 = 0$.

- $\kappa < \alpha < \kappa^+$: We'll distinguish between α a successor ordinal and a limit ordinal. Furthermore, we'll assume that $X_{\beta,n}$ has been defined for all ordinals β smaller than α.

  If α is a successor ordinal such that $\alpha = \beta + 1$ we define $X_{\alpha,n}$ as:

  - $X_{\alpha,0} = \{\beta\}$
  - $X_{\alpha,n+1} = X_{\beta,n}$

  Now assume it holds for α, so $\bigcup_{n<\omega} X_{\alpha,n} = \alpha$ and for all n we have $\langle X_{\alpha,n}\rangle \leq \kappa^n$ (where $\langle X\rangle$ is the order type of X). Now $\bigcup_{n<\omega} X_{\alpha+1,n} = \{\alpha\} \cup \bigcup_{n<\omega} X_{\alpha,n} =_{IH} \alpha \cup \{\alpha\} = \alpha + 1$. Now $X_{\alpha+1,0} = \{\alpha\}$ which has order type 1, so $1 \leq \kappa^0$. Furthermore, we have $\langle X_{\alpha+1,n+1}\rangle = \langle X_{\alpha,n}\rangle \leq_{IH} \kappa^n \leq \kappa^{n+1}$.

  Now let α be a limit ordinal. Then we know that $\alpha = \bigcup_{\beta<\alpha} \beta$. So by assumption $\bigcup_{\beta<\alpha} = \bigcup_{\beta<\alpha} \bigcup_{n<\omega} X_{\beta,n} = \bigcup_{n<\omega} \bigcup_{\beta<\alpha} X_{\beta,n}$. And therefore we know that $\bigcup_{\beta<\alpha} X_{\beta,n} \leq \alpha$. And as $cf\alpha \leq \kappa$ it follows that $cf(\bigcup_{\beta<\alpha} X_{\beta,n}) \leq \kappa$. Now let us say $\epsilon = cf(\bigcup_{\beta<\alpha} X_{\beta,n})$, so $\epsilon \leq \kappa$. So there is some $X_{\gamma,n}$ of cardinality ϵ that is cofinal in $\bigcup_{\beta\leq\alpha} X_{\beta,n}$. So now let $X_{\alpha,n+1}$ be the union of all these $X_{\gamma,n}$. Then we see that $\bigcup_{n<\omega} X_{\alpha,n} = \alpha$. And now as each $\bigcup_{\beta<\alpha} X_{\beta,n}$ had order type of at most κ^n we see that $X_{\alpha,n+1}$ has order type at most $\kappa^n \cdot \epsilon$. And as $\epsilon \leq \kappa$ we get $\kappa^n \cdot \epsilon \leq \kappa^n \cdot \kappa = \kappa^{n+1}$

□

Exercise 28

Proposition. *Let λ be an infinite cardinal and $\langle \kappa_i; i < \lambda \rangle$ an increasing sequence of regular cardinals, and $\kappa = sup_{i<\lambda}\kappa_i$. Then $2^\kappa = \prod_{i<\lambda} 2^{\kappa_i}$.*

Proof. We will prove this by giving a bijection between the set $P(\kappa)$ and the product set $\prod_{i<\lambda} P(\kappa_i)$.

First consider $\sum_{i<\lambda} \kappa_i = \lambda \cdot \sup_{i<\lambda} \kappa_i = \lambda \cdot \kappa$. Now we see that $\lambda \leq \kappa$ as $\kappa_i < \kappa$ and the sequence is increasing. Hence, $\sum_{i<\lambda} \kappa_i = \kappa$. Therefore we see that there is a bijection $f : \kappa \to \sum_{i<\lambda} \kappa_i$. So we define the function $g : P(\kappa) \to P(\sum_{i<\lambda} \kappa_i)$ so that $g(s) = f[s]$. This clearly is a bijection as f was a bijection.

Now we know that we can see $\prod_{i<\lambda} P(\kappa_i)$ as the set of choice functions on $P(\kappa_i)$. So let $h : P(\sum_{i<\lambda} \kappa_i) \to \prod_{i<\lambda} P(\kappa_i)$ be such that $h(s') = c$ where c is the choice function such that for all i we have $c(i) = t$ iff $s' \cap \kappa_i = t$. This also is a bijection.

So now we can see that $h \circ g$ is a bijection between $P(\kappa)$ and $\prod_{i<\lambda} P(\kappa_i)$. □

Exercise 29

a)

Proposition. $\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}$

Proof. First note that by Theorem 5.20i) we have $\aleph_1^{\aleph_1} = 2^{\aleph_1}$. Now we can distinguish two cases:

- $\aleph_\omega > \aleph_1^{\aleph_1}$: Now let $\mu < \aleph_\omega$, so $\mu = \aleph_{n+1}$ for some $n < \omega$. By the Hausdorff formula we see that $\aleph_{n+1}^{\aleph_1} = \aleph_n^{\aleph_1} \cdot \aleph_{n+1}$. We can continue this progress until we get to $\aleph_{n+1}^{\aleph_1} = \aleph_1^{\aleph_1} \cdot \ldots \cdot \aleph_{n+1}$. And as we know of each of these that they are strictly smaller than $\aleph_\omega$ we see that $\aleph_{n+1}^{\aleph_1} < \aleph_\omega$. Therefore we see that for all $\mu < \aleph_\omega$ we have that $\mu^{\aleph_1} < \aleph_\omega$. And as $cf\aleph_\omega = \aleph_0 \leq \aleph_1$ we see by Theorem 5.20 iii)b) that $\aleph_\omega^{\aleph_1} = \aleph_\omega^{cf\aleph_\omega} = \aleph_\omega^{\aleph_0}$. And as $\aleph_\omega > \aleph_1^{\aleph_1}$ we also have $\aleph_\omega^{\aleph_0} > \aleph_1^{\aleph^1}$ and therefore $\aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1} = \aleph_\omega^{\aleph_0} = \aleph_\omega^{\aleph_1}$.
- $\aleph_\omega \leq \aleph_1^{\aleph_1}$: Then as $\aleph_1 < \aleph_\omega$ and $\aleph_1^{\aleph_1} \geq \aleph_\omega$ we get by Theorem 5.20 ii) that $\aleph_\omega^{\aleph_1} = \aleph_1^{\aleph_1}$. So we see that $\aleph_\omega^{\aleph_0} < \aleph_\omega^{\aleph_1} = \aleph_1^{\aleph_1}$. And therefore $\aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1} = 2^{\aleph_1} = \aleph_\omega^{\aleph_1}$.

□

b)

Proposition. *If $2^{\aleph_1} = \aleph_2$ and $\aleph_\omega^{\aleph_0} > \aleph_{\omega_1}$, then $\aleph_{\omega_1}^{\aleph_1} = \aleph_\omega^{\aleph_0}$.*

Proof. As $\aleph_\omega < \aleph_{\omega_1}$ and $\aleph_\omega^{\aleph_1} \geq \aleph_\omega^{\aleph_0} > \aleph_{\omega_1}$, we see by Theorem 5.20 ii) that $\aleph_{\omega_1}^{\aleph_1} = \aleph_\omega^{\aleph_1}$. Now by a) we see that $\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}$. And by our assumption this means $\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot \aleph_2$. So as $\aleph_2 < \aleph_\omega \leq \aleph_\omega^{\aleph_0}$ we see that $\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0}$. And therefore we have now shown that $\aleph_{\omega_1}^{\aleph_1} = \aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0}$. □

c)

Proposition. *If $2^{\aleph_0} \geq \aleph_{\omega_1}$, then $\gimel(\aleph_\omega) = 2^{\aleph_0}$ and $\gimel(\aleph_{\omega_1}) = 2^{\aleph_1}$.*

Proof. So first let us consider $\gimel(\aleph_\omega)$. We know that $cf\aleph_\omega = \aleph_0$, and thus $\gimel(\aleph_\omega) = \aleph_\omega^{cf\aleph_\omega} = \aleph_\omega^{\aleph_0}$. Furthermore, we know that $2 < \aleph_\omega$ and $2^{\aleph_0} \geq \aleph_{\omega_1} > \aleph_\omega$. And therefore we see by Theorem 5.20 ii) that $\aleph_\omega^{\aleph_0} = 2^{\aleph_0}$. So indeed $\gimel(\aleph_\omega) = 2^{\aleph_0}$.

Now let us consider $\gimel(\aleph_{\omega_1})$. We know that $cf\aleph_{\omega_1} = \aleph_1$, and thus $\gimel(\aleph_{\omega_1}) = \aleph_{\omega_1}^{\aleph_1}$. Furthermore, we know that $2 < \aleph_{\omega_1}$ and $2^{\aleph_1} \geq 2^{\aleph_0} \geq \aleph_{\omega_1}$. So by Theorem 5.20 ii) we see that $\aleph_{\omega_1}^{\aleph_1} = 2^{\aleph_1}$. And so also $\gimel(\aleph_{\omega_1}) = 2^{\aleph_1}$. □

Exercise 30

Proposition. *If β is such that $2^{\aleph_\alpha} = \aleph_{\alpha+\beta}$ for all α. Then $\beta < \omega$.*

Proof. So suppose this is not the case, so $\beta \geq \omega$. Now let α be minimal such that $\alpha + \beta > \beta$. Now we can write β in its Cantor Normal form and say that ω^γ is the biggest factor in it. For contradiction suppose that α is a successor cardinal so that $\alpha = \delta + 1$. As α was minimal $\delta + \beta \leq \beta$ and thus $\delta < \omega^\gamma$. But as ω^γ is a limit ordinal we also have $\alpha < \omega^\gamma$. So then we α gets swallowed by β. And thus $\alpha + \beta \leq \beta$, a contradiction. So we see that α is a limit.

Now let $\kappa = \aleph_{\alpha+\alpha}$. As α is a limit ordinal, so is $\alpha + \alpha$. So then κ is a limit ordinal, i.e., $\kappa = \bigcup_{\epsilon<\alpha} \aleph_{\alpha+\epsilon}$. This is a union of α elements so $cf\kappa \leq \alpha < \aleph_{\alpha+\alpha} = \kappa$. And therefore we see that κ is singular.

Now we'll assume that for all $\epsilon < \alpha$ we have $2^{\aleph_{\alpha+\epsilon}} = \aleph_{\alpha+\beta}$. So as κ is a limit, we see by Theorem 5.16 iii) that $2^\kappa = (2^{<\kappa})^{cf\kappa}$. Now as $2^{<\kappa} = \sup_{\epsilon<\alpha} \aleph_{\alpha+\epsilon} = \aleph_{\alpha+\beta}$ we see that $2^\kappa \leq (\aleph_{\alpha+\beta})^{cf\kappa} = (2^{\aleph_\alpha})^{cf\kappa} \leq (2^{\aleph_\alpha})^\alpha = 2^{\aleph_\alpha} = \aleph_{\alpha+\beta}$. But by our assumption we have that $2^\kappa = 2^{\aleph_{\alpha+\alpha}} = \aleph_{\alpha+\alpha+\beta}$. And that is strictly bigger than $\aleph_{\alpha+\beta}$ as already $\alpha + \beta > \beta$. So we have a contradiction and we find that indeed $\beta < \omega$. □