Homework Sheet #7

MasterMath: Set Theory
2021/22: 1st Semester
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Deadline for Homework Set #7: Monday, 1 November 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(24) Let κ, λ, μ, and ν be non-zero cardinals. Show the following rules of *cardinal arithmetic* by providing explicit bijections or injections between the corresponding sets:

(a) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$,

(b) $\kappa^\lambda \cdot \kappa^\mu = \kappa^{\lambda+\mu}$, and

(c) $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.

(d) If $\kappa \leq \lambda$ and $\mu \leq \nu$, then $\kappa^\mu \leq \lambda^\nu$.

(25) Show that the following sets have cardinality $2^{\aleph_0}$:

(a) $\mathbb{R}$,

(b) $\mathbb{C}$,

(c) the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

[*Hint.* For (a) and (b), first be precise about what $\mathbb{R}$ and $\mathbb{C}$ actually are. You might wish to use the Cantor-Schröder-Bernstein Theorem. For (c), note that continuous functions that agree on the rationals are equal (why?).]

(26) Show the following additional closure properties of limit levels of the von Neumann hierarchy.

(a) Let λ be any limit ordinal. Assume ZFC and show that $\mathbf{V}_\lambda$ satisfies AC by showing that the choice function for any set $x \in \mathbf{V}_\lambda$ lies in $\mathbf{V}_\lambda$.

(b) Let λ be any uncountable limit *cardinal.* Assume $\mathsf{ZFC} + \mathsf{GCH}$ and show that $\mathbf{V}_\lambda$ satisfies GCH by showing that for each $\kappa < \lambda$, the bijection between κ^+ and the power set of κ is in $\mathbf{V}_\lambda$.