(24) Let κ, λ, μ, and ν be nonzero cardinals.

(a) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$

Proof. It suffices to show that $|\mathbf{Set}(\mu, \kappa \times \lambda)| = |\mathbf{Set}(\mu, \kappa) \times \mathbf{Set}(\mu, \lambda)|$. Define G and H such that

$$\mathbf{Set}(\mu, \kappa \times \lambda) \underset{H}{\overset{G}{\rightleftarrows}} \mathbf{Set}(\mu, \kappa) \times \mathbf{Set}(\mu, \lambda)$$

where $G(f) = (\pi_\kappa \circ f, \pi_\lambda \circ f)$ and $H(f_\kappa, f_\lambda) = (x \mapsto (f_\kappa(x), f_\lambda(x)))$. Then H is the inverse of G, so G is a bijection. $\square$

(b) $\kappa^\lambda \cdot \kappa^\mu = \kappa^{\lambda+\mu}$

Proof. Define G and H such that

$$\mathbf{Set}(\lambda, \kappa) \times \mathbf{Set}(\mu, \kappa) \underset{H}{\overset{G}{\rightleftarrows}} \mathbf{Set}(\lambda \sqcup \mu, \kappa)$$

where

$$\begin{aligned} G(f_\lambda, f_\mu) &= (x, i) \mapsto \begin{cases} f_\lambda(x) & \text{if } i = 0 \\ f_\mu(x) & \text{if } i = 1, \text{ and} \end{cases} \\ H(f) &= (f \restriction \{(x, i) \mid i = 0\}, f \restriction \{(x, i) \mid i = 1\}). \end{aligned}$$

Then H is the inverse of G, so G is a bijection. $\square$

(c) $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$

Proof. This follows from the fact that, for any two given sets X and Y, the set $\mathbf{Set}(X, Y) = Y^X$ is an exponential object of X and Y in **Set**. In other words, Currying and unCurrying of functions witness our desired bijection. $\square$

(d) If $\kappa \leq \lambda$ and $\mu \leq \nu$, then $\kappa^\mu \leq \lambda^\mu$.

Proof. Since $\kappa \leq \lambda$ and $\mu \leq \nu$, we know that $\kappa \subseteq \lambda$ and $\mu \subseteq \nu$. Define $I : \mathbf{Set}(\mu, \kappa) \to \mathbf{Set}(\nu, \lambda)$ by the equation

$$I(f)(x) = \begin{cases} f(x) & \text{if } x \in \mu \\ 0 & \text{otherwise.} \end{cases}$$

Then $I(f) : \nu \to \lambda$ and, clearly, I is injective, so I witnesses $|\mathbf{Set}(\mu, \kappa)| \leq |\mathbf{Set}(\nu, \lambda)|$. $\square$

(25) The following sets have cardinality $2^{\aleph_0}$.

(a) $\mathbb{R}$.

Proof. $|\mathbb{R}| = |[0,1]| = |\mathscr{P}(\omega)| = |\mathbf{Set}(\omega, \{0,1\})| = 2^{\aleph_0}$. □

(b) $\mathbb{C}$

Proof. $|\mathbb{C}| = |\mathbb{R}^2| = |\mathbb{R}| = 2^{\aleph_0}$. □

(c) $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$

Proof. By ZFC, we know that sequential continuity is equivalent to continuity. Because every real r is a limit of a convergent sequence of rational numbers $\langle r_i \mid i \in \omega \rangle$, we know that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and agree on rationals, then $f = g$ because, for all r, $f(r) = \lim_{i\to\infty} f(r_i) = \lim_{i\to\infty} g(r_i) = g(r)$.

Define the function $G : \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\} \to \mathbf{Set}(\mathbb{Q}, \mathbb{R})$ by the equation $G(f) = f \restriction \mathbb{Q}$. By the above, G is injective, so $|\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}| \leq |\mathbf{Set}(\mathbb{Q}, \mathbb{R})| = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$. □

(26) The von Neumann hierarchy has the following closure properties.

(a) Assume $V \models \mathsf{ZFC}$ and $\lambda \in V$ is a limit ordinal. Then $V_\lambda \models \mathsf{AC}$.

Proof. Suppose that $x \in V_\lambda$ and that $f : x \to \bigcup x$ is a choice function. Since λ is a limit ordinal and $x \in V_\lambda$, we know $\varrho(x) = \alpha < \lambda$. This implies that $\varrho(\bigcup x) \leq \alpha$, and so $f \in \mathscr{P}(x \times \bigcup x) \in V_{\alpha+4}$. Thus, $\varrho(f) \leq \alpha + 3 < \lambda$, so $f \in V_\lambda$. Thus $V_\lambda \models \mathsf{AC}$. □

(b) Assume $V \models \mathsf{ZFC} + \mathsf{GCH}$ and that $\lambda \in V$ is an uncountable limit cardinal. Then $V_\lambda \models \mathsf{GCH}$.

Proof. If λ is a limit cardinal then, for any cardinal $\kappa < \lambda$ we have $\kappa^+ < \lambda$. This implies that κ^+ and $\mathscr{P}(\kappa)$ are elements of V_λ, which, in turn, implies that $\mathbf{Set}(\kappa^+, \mathscr{P}(\kappa))$ is an element of V_λ.

Since $V \models \mathsf{GCH}$, we know there is some $f \in V$ such that $f : \kappa^+ \to \mathscr{P}(\kappa)$ is a bijection. By the above, $\varrho(f) < \lambda$, so $f \in V_\lambda$, which implies that $V_\lambda \models \mathsf{GCH}$. □