Homework Sheet #6

MasterMath: Set Theory
2021/22: 1st Semester
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Deadline for Homework Set #6: Monday, 25 October 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(21) Prove the *Knaster-Tarski Fixed Point Theorem*: Let X be a set and $F : \mathrm{P}(X) \to \mathrm{P}(X)$ a $\subseteq$-monotone function, i.e., if $A \subseteq B$, then $F(A) \subseteq F(B)$. Then F has a fixed point, i.e., a set $A \subseteq X$ such that $A = F(A)$.

Use the Knaster-Tarski Fixed Point Theorem to prove the *Banach Decomposition Theorem*: Let X and Y be sets and $f : X \to Y$ and $g : Y \to X$ arbitrary functions. Then there are disjoint decompositions $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ such that $f[X_1] = Y_1$ and $g[Y_2] = X_2$.

[*Hint.* Apply Knaster-Tarski to $F(S) := X \backslash g[Y \backslash f[S]]$.]

Finally, derive the Cantor-Schröder-Bernstein Theorem from the Banach Decomposition Theorem.

(22) Let X be a set of pairwise disjoint non-empty sets, i.e., if $x, x' \in X$, then $x \neq \varnothing \neq x'$ and $x \cap x' = \varnothing$. We say that C is a *choice set for* X if for each $x \in X$, the set $x \cap C$ has exactly one element. The *Axiom of Choice Sets* says that every set of pairwise disjoint, non-empty sets has a choice set.

Show that (on the basis of the axioms of ZF), the Axiom of Choice and the Axiom of Choice Sets are equivalent.

Why can you not get rid of the requirement that the sets in X are pairwise disjoint?

(23) The following is an excerpt from Jech's book (p. 49):

> In algebra and point set topology, one often uses the following version of the Axiom of Choice. We recall that if $(P, <)$ is a partially ordered set, then $a \in P$ is called *maximal* in P if there is no $x \in P$ such that $a < x$. If X is a nonempty subset of P, then $c \in P$ is an *upper bound* of X if $x \leq c$ for every $x \in X$.
>
> We say that a nonempty $C \subset P$ is a *chain* in P if C is linearly ordered by $<$.
>
> **Theorem 5.4 (Zorn's Lemma).** *If $(P, <)$ is a nonempty partially ordered set such that every chain in P has an upper bound, then P has a maximal element.*

Prove Zorn's Lemma in ZFC. Be very explicit about your use of the Axiom of Choice.