Exercise 21

Prove the Knaster-Tarski Fixed Point Theorem, the Banach decomposition Theorem and derive the cantor-Schröder-Bernstein Theorem.

Knaster-Tarski Fixed Point Theorem (KT-FPT): Let X be a set and $F : P(X) \to P(X)$ a $\subseteq$-monotone function. Then F has a fixed point.

Banach Decomposition theorem (BDT): Let X and Y be sets and $f : X \to Y$ and $g : Y \to X$ be arbitrary functions. Then there are disjoint decompositions $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ such that $f[X_1] = Y_1$ and $f[Y_2] = X_2$.

Cantor-Schröder-Bernstein Theorem (CSBT): Let X and Y be sets. If there is an injection from X to Y and one from Y to X, then there exists a bijection between X and Y.

We start by proving KI-FPT. Let X be a set and $F : P(X) \to P(X)$ a $\subseteq$-monotone function. Define the following set:

$$\mathcal{A} := \{A \subseteq X \mid A \subseteq F(A)\}$$

Consider the following set: $B = \bigcup \mathcal{A}$. We will show $B = F(B)$ by showing two inclusions:

- $B \subseteq F(B)$

  Let $b \in B$. Then there is $A \in \mathcal{A}$ such that $b \in A$. As $A \subseteq F(A)$, we see $b \in F(A)$. By monotonicity of F and $A \subseteq B$, we see $F(A) \subseteq F(B)$. So $b \in F(B)$. Thus, $B \subseteq F(B)$.

- $F(B) \subseteq B$

  As $B \subseteq F(B)$, $F(B) \subseteq F(F(B))$ by monotonicity. Thus, $F(B) \in \mathcal{A}$ an so, $F(B) \subseteq B$.

We conclude that B is a fixed point of F.

Now, we prove BDT. Let X and Y be sets and $f : X \to Y$ and $g : Y \to X$ functions. Consider the following function:

$$F : P(X) \to P(X) : S \mapsto X \setminus g[Y \setminus f[S]]$$

We show this function is monotone: Let A and B be X-subsets such that $A \subseteq B$, then:

$$\begin{aligned} A \subseteq B &\to f[A] \subseteq f[B] \\ &\to Y \setminus f[B] \subseteq Y \setminus f[A] \\ &\to g[Y \setminus f[B]] \subseteq g[Y \setminus f[A]] \\ &\to X \setminus g[Y \setminus f[A]] \subseteq X \setminus g[Y \setminus f[B]] \\ &\to F(A) \subseteq F(B) \end{aligned}$$

So indeed, F is monotone. By KT-FPT, this function has a fixed point C:

$$C = F(C) = X \setminus g[Y \setminus f[C]]$$

Then, also: $X = C \cup g[Y \setminus f[C]]$ and $Y = f[C] \cup (Y \setminus f[C])$. So, we choose $X_1 = C$, $X_2 = g[Y \setminus f[C]]$, $Y_1 = f[C]$ and $Y_2 = Y \setminus f[C]$ (note these are disjoint). This proves BDT.

Finally, we will derive CSBT. Let X and Y be sets and $f : X \to Y$ and $g : Y \to X$ be injective functions. By BDT, there are disjoint compositions $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ such that $f[X_1] = Y_1$ and $g[Y_2] = X_2$. As f and g are injections, so are $f \restriction X_1$ and $g \restriction Y_2$. Note that these functions are reversible when domain and range are restricted: f acts as a bijection between X_1 and Y_1 and g acts as a bijection between Y_2 and X_2.

Now, consider the following function:

$$F : X \to Y : x \mapsto \begin{cases} f(x) & \text{if } x \in X_1 \\ g^{-1}(x) & \text{if } x \in X_2 \end{cases}$$

This function is one-to-one, as f and g^{-1} are. If $y \in Y$, then either $y \in Y_1$ or Y_2. In the former case, we know there is $x \in X_1$ such that $f(x) = y$, as $f[X_1] = Y_1$. In the latter case, we know there is a unique x such that $g(y) = x$. Thus, $g^{-1}(x) = y$. Thus, F is onto. We conclude F is a bijection. This proves CSBT.

22) X set of pairwise disj. non-empty sets. $\mathcal{C}$ is a CHOICE SET for X iff $\forall x \in X$, $|x \cap \mathcal{C}| = 1$

(ACS): Forall X as above, there exists a choice set

To show: In ZF, (AC) $\Leftrightarrow$ (ACS)

Proof. "$\Rightarrow$" Let X be a set of pairw. disj. non-empty sets. By assumption, $\emptyset \notin X$. By (AC), there exists a choice function f s.t. $\forall x \in X$ $f(x) \in x$. We then consider as our candidate choice set $\mathcal{C} = f(X)$.

Forall $x \in X$ we have $f(X) \cap x \neq \emptyset$, since $f(x) \in x$.

Suppose by contradiction that $f(X) \cap x \supsetneq \{f(x)\}$, i.e. there exists an $z \in f(X) \cap x$, $z \neq f(x)$ (1). Since $z \in f(X)$ $z = f(y)$ for some $y \in X$. But then we know that $f(y) \in y$ (2) since f is a choice function. By (1) and (2) we get $f(y) \in x \cap y$, which is a contradiction since by assumption $\forall x, y \in X$, $x \cap y = \emptyset$

"$\Leftarrow$" Let S be a family of nonempty sets. Consider the set $X = \{\{x\} \times x,\ x \in S\}$. Since $x \neq \emptyset$ $\forall x \in S$, $\emptyset \notin X$. Also, the elements of X are pairwise disjoint: let $\{x\} \times x$, $\{y\} \times y \in X$, then $\{x\} \times x \cap \{y\} \times y = \emptyset$ since $x \neq y$ $\forall x, y \in S$ (by extensionality). We then apply (ACS) to X: $\exists$ a choice set $\mathcal{C}$ s.t. $\forall \{x\} \times x \in X$ $|\mathcal{C} \cap \{x\} \times x| = 1$, furthermore $\mathcal{C} \cap \{x\} \times x = (x, x')$ where $x' \in x$. Then define the map f s.t. $\forall x \in S$ $f(x) = x'$, where $x' = \mathcal{C} \cap \{x\} \times x$. By construction, f is functional and it is a choice function for S.

23 ~~[illegible]~~ : Prove Zorn's lemma in ZFC.
If $(P, <)$ is a partially ordered set such that each chain in nonempty P has an upper bound, then P has a maximal elt.

Let $(P, <)$ be a non-empty partially ordered set s.t. each chain C has an upper bound:
$u \in C$ s.t. $c \leq u \quad \forall c \in C$.

Now, define by transfinite recursion a sequence as follows:

Let c be the choice function.
Def $a_0 := c(P) \in P$ (application AC)
and $P_{a_0} := \{ x \in P \mid a_0 < x \}$
If $P_{a_0} = \emptyset$, a_0 is maximal and we are done.
If $P_{a_0} \neq \emptyset$, def $a_1 := c(P_{a_0}) \in P_{a_0}$ and so:

$a_0 := c(P) \in P$ (Application AC)

$a_{n+1} := c(P_{a_n}) \in P_{a_n}$ "

and $a_\lambda := c(\bigcap_{\gamma<\lambda} P_\gamma) \in \bigcap_{\gamma<\lambda} P_\gamma$ for λ limit and $\gamma < \lambda$.

If P has no maximal elt, this definition would define a function from the ordinals to P that is injective.

$$\left(f : \begin{array}{l} 0 \longmapsto c(P) \\ n+1 \longmapsto c(P_{a_n}) \\ \lambda \longmapsto c(\bigcap_{\gamma<\lambda} P_\gamma) \end{array} \right)$$

This is in contradiction with Hartogs's theorem, which says that for each set P there exists an ordinal that does not inject into P, so P must have a maximal elt. $\square$