(17) $F(\beta,\alpha) := \{ f:\beta\to\alpha : \text{ for all but finitely many } \gamma\in\beta \ f(\gamma)=0\}$.

$f < g \Leftrightarrow f(\mu) < g(\mu) \quad \mu := \max\{\gamma\in\beta ; f(\gamma)\neq g(\gamma)\}$

To show: $(F(\alpha,\beta), <) \cong (\alpha^{\beta}, \in)$.

- Let for any $f\in F(\beta,\alpha)$

  $\gamma_1 := \max\{\gamma\in\beta ; f(\gamma_1)\neq 0\}$

  $\gamma_2 := \max\{\gamma\in\beta, \gamma\neq\gamma_1 \text{ and } f(\gamma_2)\neq 0\}$

  and so forth.

~~Now consider the functions~~ By definition, there are only finitely many $\gamma\in\beta$ s.t. $f(\gamma)\neq 0$, so for any f one obtains a finite sequence $\gamma_1, \gamma_2, .. \gamma_n$ ~~etc~~.

Now consider the function

$h: F(\beta,\alpha)\to\alpha^{\beta}$ given by

$h(f) = \alpha^{\gamma_1} f(\gamma_1) + \alpha^{\gamma_2}\cdot f(\gamma_2) + .. + \alpha^{\gamma_n} f(\gamma_n)$

I claim this function is order preserving and bijective.

– Order preserving: Suppose $f<g$, so $f(\mu)<g(\mu)$.

Options:

- $f(\mu)=0$. Then as $h(g)$ has a factor $\alpha^{\mu}\cdot g(\mu)$ and f does not, as all larger factors are the same $h(f)\in h(g)$.
- $f(\mu)\neq 0$. Then $\alpha^{\mu}\cdot f(\mu) \in \alpha^{\mu} g(\mu)$ as $f(\mu)<g(\mu)$, so as all larger factors are the same $h(f)\in h(g)$.

  (Suppose $\mu=\gamma_k$. then $\alpha^{\gamma_k} f(\gamma_k) + .. + \alpha^{\gamma_n} f(\gamma_n) \in \alpha^{\gamma_k} g(\gamma_k)$).

– It also follows from the above that h is injective, As if $f\neq g$, then w.l.o.g $f<g$ so $h(f)\in h(g)$ and $h(f)\neq h(g)$.

– Surjective: Take any $x\in\alpha^{\beta}$, we can write it in Cantor normal form: (thm 2.26 Jech)

Use base α

$x = \alpha^{\gamma_1}\cdot k_1 + ... + \alpha^{\gamma_n}\cdot k_n$ *

Now take $f:\beta\to\alpha$ given by $f(\gamma_i)=k_i \ \forall 1\leq i\leq n$ and $f(\gamma)=0$ if $\gamma\neq\gamma_i$ for some i. Clearly $h(f)=x$.

- Thus $h: F(\alpha,\beta)\to\alpha^{\beta}$ is bijective and order preserving. Thus $(F(\alpha,\beta),<)\cong(\alpha^{\beta},\in)$.

* Recall that we have shown in GI 3 that we can use base α

Exercise 18

We'll determine the rank of $P(x)$, $\bigcup x$, $\{x,y\}$ and $x \times y$.

Proposition. *Let $\varrho(x) = \alpha$. Then $\varrho(P(x)) = \alpha + 1$.*

Proof. As $\varrho(x) = \alpha$ we know x first appears in $V_{\alpha+1}$. So all elements of x are in V_α, and thus all subsets of x are in $V_{\alpha+1}$. Some might have been there before, but at least x itself was not. So then we see that $P(x)$ first appears in $V_{\alpha+2}$. Hence, $\varrho(P(x)) = \alpha + 1$. $\square$

Proposition. *Let $\varrho(x) = \alpha$. Then $\varrho(\bigcup x) = \alpha$ if α a limit ordinal and $\varrho(\bigcup x) = \alpha - 1$ otherwise.*

Proof. Suppose $\alpha = 0$, so $\varrho(x) = 0$. Then $x \in V_1$, and thus $x = \emptyset$. But then $\bigcup x = \emptyset$. So also $\varrho(\bigcup x) = 0$.

So now suppose $\alpha > 0$ is a successor ordinal. Then say $\alpha = \delta + 1$. All elements of x first appear together in V_α. So all their elements first appear together in V_δ. So then clearly $\bigcup x$ first appears in $V_{\delta+1} = V_\alpha$. Hence, $\varrho(\bigcup x) = \delta = \alpha - 1$.

Now suppose that $\alpha > 0$ is a limit ordinal. Then x first appears in $V_{\alpha+1}$, and thus all elements of x first appear in V_α. So there is no no $\beta < \alpha$ such that all elements of elements of x are in V_β. For otherwise, all elements of x would be in $V_{\beta+1}$ and thus x would be in $V_{\beta+2}$. Hence, all elements of elements also first appear in V_α. So we see that $\bigcup x$ first appears in $V_{\alpha+1}$, i.e. $\varrho(\bigcup x) = \alpha$. $\square$

Proposition. *Let $\varrho(x) = \alpha, \varrho(y) = \beta$. Then $\varrho(\{x,y\}) = max(\alpha, \beta) + 1$.*

Proof. As $\varrho(x) = \alpha$ and $\varrho(y) = \beta$ we know that $\delta = max(\alpha, \beta)$ is the least ordinal such that $x, y \in V_{\delta+1}$. So then by construction $\{x,y\}$ first appears in $V_{\delta+2}$. Hence $\varrho(\{x,y\}) = \delta + 1 = max(\alpha, \beta) + 1$. $\square$

Proposition. *Let $\varrho(x) = \alpha, \varrho(y) = \beta$. Then $\varrho(x \times y) = max(\alpha, \beta) + 2$.*

Proof. By the lecture notes we know that $x \times y \subseteq P(P(\bigcup\{x,y\}))$. By previous results we know that $\varrho(\{x,y\}) = max(\alpha, \beta) + 1$. And as that is not zero we see that $\varrho(\bigcup\{x,y\}) = max(\alpha, \beta)$. And thus $\varrho(P(P(\bigcup\{x,y\}))) = max(\alpha, \beta) + 2$. So $P(P(\bigcup\{x,y\}))$ first appears in $V_{max(\alpha,\beta)+3}$, and thus all of its elements first appear together in $V_{max(\alpha,\beta)+2}$. In particular, all elements of the form $\{a, \{a,b\}\}$ for $a \in x$ and $b \in y$ first appear in $V_{max(\alpha,\beta)+2}$. And thus $x \times y$ first appears in $V_{max(\alpha,\beta)+3}$, i.e. $\varrho(x \times y) = max(\alpha, \beta) + 2$. $\square$

Exercise 19

Proposition. *Let α be an ordinal. Then $(V_\alpha, \in)$ is a model of the axioms of Extensionality and Foundation.*

Proof. We work in the axiom system ZF. Let α be some ordinal.

- *Extensionality*: It is to show that $(V_\alpha, \in) \vDash Ext$, so that $\forall x, y(\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x = y)$ holds relativized over V_α. In other words, it should hold that $\forall x, y \in V_\alpha(\forall z \in V_\alpha(z \in x \leftrightarrow z \in y) \leftrightarrow x = y)$. So suppose that it does not hold, then there are $x, y \in V_\alpha$ such that $\forall z \in V_\alpha(z \in x \leftrightarrow z \in y)$ but $x \neq y$. So by Extensionality we know that without loss of generality, there is some z such that $z \in x$ and $z \notin y$. By construction of the hierarchy we know that all elements of x appear in some V_β with $\beta \leq \alpha$. So then by a lemma from the lecture we see that $V_\beta \subseteq V_\alpha$. So there is some V_β such that $z \in V_\beta$ and $\beta \leq \alpha$, and thus $z \in V_\alpha$. But this is in contradiction with $\forall z \in V_\alpha(z \in x \leftrightarrow z \in y)$. Hence, $(V_\alpha, \in) \vDash Ext$.

- *Foundation*: It is to show that $\forall x \in V_\alpha(x \neq \emptyset \rightarrow \exists m \in V_\alpha(m \in x \wedge m \cap x = \emptyset))$. Suppose this fails, then there is a non-empty $x \in V_\alpha$ such that $\forall m \in V_\alpha$ with $m \in x$ we have $m \cap x \neq \emptyset$. As $x \in V_\alpha$ we see by transitivity of V_α that also the elements of x are in V_α. So by Foundation we know there is some $m \in x$ with $m \cap x = \emptyset$. But then we see that $m \in V_\alpha$, so we actually do have an $m \in V_\alpha$ such that $m \in x \wedge m \cap x = \emptyset$. A contradiction, so we see that indeed $(V_\alpha, \in) \vDash Foundation$.

$\square$

Exercise 20

Proposition. *For α an ordinal, the following are equivalent:*

i) For all $\beta < \alpha$, there is no bijection between α and β.

ii) For all $\beta < \alpha$, there is no injection from α into β.

iii) For all $\beta < \alpha$, there is no surjection from β onto α.

Proof. Several cases are trivial. If there is no surjection from β onto α, or no injection from α into β, then clearly there is no bijection between them. So we have $ii) \Rightarrow i)$ and $iii) \Rightarrow i)$. Now we'll show $ii) \Rightarrow iii)$ and $i) \Rightarrow ii)$ and we are done.

For a formal proof, think about how we can do this picking in ZF.

- $ii) \Rightarrow iii)$: Suppose $iii)$ does not hold. Then there is some $\beta < \alpha$ such that there is a surjection f from β onto α. So for each $x \in \alpha$ there is a $y \in \beta$ such that $f(y) = x$. Note that there might be more than one such y. Now define $g : \alpha \to \beta$ as $f(y) \mapsto y$ for $y \in \beta$. This might not be functional, so we let g' be similar to g, but we pick just one element to map to for each $x \in \alpha$. This is well-defined as f was surjective. So now we see that we have an injection from α into β. Hence, $ii)$ does not hold.

- $i) \Rightarrow ii)$: Now suppose that $ii)$ does not hold, so there is some $\beta < \alpha$ such that there is an injection $f : \alpha \to \beta$. Let $r := ran(f)$, so then $r \subseteq \beta$. As β is wellordered by $\in$, so is r. Hence, by the Representation Theorem there is some ordinal δ such that $(r, \in) \cong (\delta, \in)$. Clearly, $\delta < \beta$. And as $f : \alpha \to \beta$ is an injection, $f : \alpha \to r$ is a bijection. So there is a bijection $g : \alpha \to \delta$. Hence, $i)$ does not hold.

$\square$