Homework Sheet #4

MasterMath: Set Theory
2021/22: 1st Semester
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Deadline for Homework Set #4: Monday, 11 October 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(14) Work in FST. Let $\mathbf{X}_1 := (X_1, <_1)$ and $\mathbf{X}_2 := (X_2, <_2)$ be two strict total orders (i.e., irreflexive, transitive, and total relations). On the set $X_1 \times X_2$, we define two relations by

$$(x_1, x_2) <_{\bigcirc} (x_1', x_2') :\iff x_1 <_1 x_1' \vee (x_1 = x_1' \wedge x_2 <_2 x_2') \text{ and}$$
$$(x_1, x_2) <_{\Box} (x_1', x_2') :\iff x_1 <_1 x_1' \wedge x_2 <_2 x_2',$$

and define two product operations $\mathbf{X}_1 \otimes \mathbf{X}_2 = (X_1 \times X_2, <_{\bigcirc})$ and $\mathbf{X}_1 \boxtimes \mathbf{X}_2 = (X_1 \times X_2, <_{\Box})$.

Only one of the two produces in general a strict total order. Which one? What properties does the other one have?

In (13), you proved preservation of wellfoundedness for the operation $\oplus$. Can you show the same for the operation preserving strict total orders? If not, what are the exceptional cases?

(15) In class, we proved that in wellorders $\mathbf{W} = (W, <)$ every proper initial segment is of the form $<[w] := \{v \in W\,;\, v < w\}$ for some $w \in W$. Show that this property characterises wellorders. Please state precisely what you are proving and then provide a proof.

(16) Prove the following properties of ordinal arithmetic (in the following, α, β, and γ are ordinals):

(a) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
(b) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$.
(c) If $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$.
(d) If $\alpha < \beta$, then $\gamma + \alpha < \gamma + \beta$.
(e) If $\alpha \leq \beta$, then $\alpha \cdot \gamma \leq \beta \cdot \gamma$.
(f) If $\alpha < \beta$ and $\gamma \neq 0$, then $\gamma \cdot \alpha < \gamma \cdot \beta$.

The strict versions of (c) and (e) do not hold in general: give counterexamples.