Exercise 14.

We begin with $\otimes$. Let us see if it has the properties of strict total orders:

- Irreflexive. Let $(x_1, x_2) \in X_1 \times X_2$ and assume that $(x_1, x_2) <_\circ (x_1, x_2)$. There are two possibilites, $x_1 <_1 x_1$, absurd since $<_1$ is irreflexive and $x_1 = x_1, x_2 <_2 x_2$, absurd since $<_2$ is also irreflexive.
- Transititivy. Let $(x_1, x_2) <_\circ (x_1', x_2') <_\circ (x_1'', x_2'')$. Now we can separate in various cases:
  - $x_1 <_1 x_1' <_1 x_1''$ then by transitivity of $<_1$, $x_1 <_1 x_1''$ so $(x_1, x_2) <_\circ (x_1'', x_2'')$.
  - $x_1 <_1 x_1' = x_1''$ and $x_2' <_2 x_2''$. Then $x_1 <_1 x_1''$ so $(x_1, x_2) <_\circ (x_1'', x_2'')$.
  - $x_1 = x_1' < x_1''$ and $x_2 < x_2'$. Analogous to previous case.
  - $x_1 = x_1' = x_1''$ and $x_2 <_2 x_2' <_2 x_2''$. Then $x_1 = x_1''$ and thanks to transitivity of $<_2$ then $x_2 <_2 x_2''$ so $(x_1, x_2) <_\circ (x_1'', x_2'')$.
- Totalness. Let $(x_1, x_2), (x_1', x_2')$. Since $<_1$ is total we have three cases:
  - $x_1 <_1 x_1'$, then $(x_1, x_2) <_\circ (x_1', x_2')$.
  - $x_1' <_1 x_1$, analogous.
  - $x_1 = x_1'$, then we can again do three cases since $<_2$ is total and we will get that (x_1, x_2) and (x_1', x_2') are equal or related.

Now for $\boxtimes$ note that it is clearly irreflexive (if one of $<_1$ or $<_2$ is irreflexive). Transitivity is also true since $(x_1, x_2) <_\Box (x_1', x_2') <_\Box (x_1'', x_2'')$ so $x_1 <_1 x_1' <_1 x_1''$ so by tansitivity $x_1 <_1 x_1''$, similarly $x_2 <_2 x_2''$ so we get $(x_1, x_2) <_\Box (x_1'', x_2'')$. However we are going to check that it is not total. Consider $(\mathbb{N}, <) \boxtimes (\mathbb{N}, <)$ and let $(0, 1), (1, 0) \in \mathbb{N}$. Then $(0, 1) \not<_\Box (1, 0)$ since $1 \not< 0$ but $(1, 0) \not<_\Box (0, 1)$ by the same reason. But also $(0, 1) \neq (1, 0)$ so we conclude that $<_\Box$ is not total.

Finally let us prove that $\otimes$ preserves wellfoundedness. Let $A \subseteq X_1 \times X_2$ nonempty. Consider the set $A_1 = \{x_1 \in X_1 \mid \exists x_2.(x_1, x_2) \in A\}$, since A is not empty, neither A_1 is. And since $<_1$ is wellfounded let x_1' be a minimal element for A_1. Now define $A_2 = \{x_2 \in X_2 \mid (x_1', x_2) \in A\}$, it is clear that it is nonempty so thanks to wellfoundedness of $<_2$ it has a minimal element, let it be x_2'. Clearly (x_1', x_2') is in A, and image that there is $(x_1'', x_2'') \in A$ such that $(x_1'', x_2'') < \circ(x_1', x_2')$, then there are two possibilities:

- $x_1'' <_1 x_1'$, impossible since clearly $x_1'' \in A_1$ and x_1' is minimal.
- $x_1'' = x_1'$ and $x_2'' < x_2'$, impossible since then $x_2'' \in A_2$ (thanks to $x_1'' = x_1'$) and x_2' is minimal.

Since both cases are impossible we conclude that there is no such that (x_1'', x_2'') and so (x_1', x_2') is minimal.

Exercise 15.

Lemma. *Let $(W, <)$ be a strict total order such that for every proper initial segment I there is a $w \in W$ with $I = <[w]$. Then $<$ is wellfounded, i.e. $(W, <)$ is a wellorder.*

Proof. We proceed by contraposition. So assume that $(W, <)$ is not a wellorder, so there is a nonempty subset $X \subseteq W$ with no minimal element. Now we consider the set $I = \{w \in W \mid \forall x \in X. w < x\}$. First note that this is an initial segment since given $y \in I$ and $z < y$ given $x \in X$, $z < y < x$ and by transitivity $z < x$, Typo: $z \in I$ so $x \in I$. It is also proper, thanks to irreflexivity and that X was not empty, so there is $x \in X$ and $x \not< x$. Now we want to show that there is no $w \in W$ such that $I = <[w]$ so assume that there is such a w a we will proof a contradiction. First we prove that $w \in X$. Clearly $\neg\exists x \in X. x < w$ (since otherwise $x \in I$ and that would imply $x < x$) so thanks to $<$ being total we have that $\forall x \in X. w \leq x$. Now by R.A. assume that $w \notin X$ then $\forall x \in X. w < x$ and so $w \in I = <[w]$, i.e. $w < w$ absurd by irreflexivity. So we conclude that $w \in X$, but as we said earlier $\neg\exists x \in X. x < w$, i.e. w is minimal contrary to the hypothesis that X does not have a minimal element. So we conclude that such a w does not exist as wanted. □

Exercise 16

In this exercise I will use several lemmas that I will prove first.

Lemma 1. *For all ordinals α, we have $0 + \alpha = \alpha$.*

Proof. We prove this by transfinite induction on α.

- Suppose $\alpha = 0$. Then we see that $0 + 0 = 0 = \alpha$.
- Now suppose it holds for α, so $0 + \alpha = \alpha$. Then $0 + s(\alpha) = s(0 + \alpha) =_{IH} s(\alpha)$.
- Suppose α is a limit ordinal and for all $\delta < \alpha$ we have $0 + \delta = \delta$. Now $0 + \alpha = \bigcup\{0 + \delta; \delta \in \alpha\} =_{IH} \bigcup\{\delta; \delta \in \alpha\} = \alpha$.

□

Lemma 2. *For all ordinals α, β, γ, we have $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.*

Proof. Let α, β be arbitrary ordinals. Then we prove the lemma by transfinite induction on γ.

- Suppose $\gamma = 0$. Then $(\alpha \cdot \beta) \cdot 0 = 0 = \alpha \cdot 0 = \alpha \cdot (\beta \cdot 0)$.
- Now suppose that it holds for γ, so $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$. Then $(\alpha \cdot \beta) \cdot s(\gamma) = (\alpha \cdot \beta) \cdot \gamma + \alpha \cdot \beta =_{IH}= \alpha \cdot (\beta \cdot \gamma) + \alpha \cdot \beta$. And by the previous lemma we see that $\alpha \cdot (\beta \cdot \gamma) + \alpha \cdot \beta = \alpha(\beta \cdot \gamma + \beta) = \alpha \cdot (\beta \cdot s(\gamma))$.
- Suppose that γ is a limit ordinal and for all $\delta < \gamma$ we have $(\alpha \cdot \beta) \cdot \delta = \alpha \cdot (\beta \cdot \delta)$. Then $(\alpha \cdot \beta) \cdot \gamma = \bigcup\{(\alpha \cdot \beta) \cdot \delta; \delta \in \gamma\} =_{IH} \bigcup\{\alpha \cdot (\beta \cdot \delta); \delta \in \gamma\} = \bigcup\{\alpha \cdot \eta; \eta \in \beta \cdot \gamma\} = \alpha \cdot (\beta \cdot \gamma)$.

□

Lemma 3. *For all ordinals α, β, γ, we have $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$*

Proof. Let α, β be arbitrary ordinals. Then we prove the lemma by transfinite induction on γ.

- Suppose $\gamma = 0$. Then $(\alpha + \beta) + 0 = \alpha + \beta = (\alpha + 0) + \beta$.
- Now suppose that it holds for γ, so $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$. Then $(\alpha + \beta) + s(\gamma) = s((\alpha + \beta) + \gamma) =_{IH} s(\alpha + (\beta + \gamma)) = \alpha + s(\beta + \gamma) = \alpha + (\beta + s(\gamma))$.

  Suppose γ is a limit ordinal and for all $\delta < \gamma$ we have $(\alpha + \beta) + \delta = \alpha + (\beta + \delta)$. Then we have $(\alpha + \beta) + \gamma = \bigcup\{(\alpha + \beta) + \delta; \delta \in \gamma\} =_{IH} \bigcup\{\alpha + (\beta + \delta); \delta \in \gamma\} = \bigcup\{\alpha + \eta; \eta \in \beta + \gamma\} = \alpha + (\beta + \gamma)$.

□

a)

Proposition. *For α, β, γ ordinals, we have $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.*

Proof. Let α, β be arbitrary ordinals and do a proof by transfinite induction on γ.

- Suppose $\gamma = 0$. Then we have $\alpha \cdot (\beta + 0) = \alpha + \beta = \alpha \cdot \beta + 0 = \alpha \cdot \beta + \alpha \cdot 0$.
- Now suppose it holds for γ, so $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$. Then we have $\alpha \cdot (\beta + s(\gamma)) = \alpha \cdot s(\beta + \gamma) = \alpha \cdot (\beta + \gamma) + \alpha =_{IH} \alpha \cdot \beta + \alpha \cdot \gamma + \alpha = \alpha \cdot \beta + \alpha \cdot s(\gamma)$.
- Suppose γ is a limit ordinal and for all $\delta < \gamma$ we have $\alpha \cdot (\beta + \delta) = \alpha \cdot \beta + \alpha \cdot \delta$. Then we have $\alpha \cdot (\beta + \gamma) = \alpha \cdot \bigcup\{\beta + \delta; \delta \in \gamma\} = \bigcup\{\alpha \cdot (\beta + \delta); \delta \in \gamma\} =_{IH} \bigcup\{\alpha \cdot \beta + \alpha \cdot \delta; \delta \in \gamma\} = \bigcup\{\alpha \cdot \beta + \eta; \eta \in \alpha \cdot \gamma\} = \alpha \cdot \beta + \alpha \cdot \gamma$

□

b)

Proposition. *For α, β, γ ordinals, we have $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$.*

Proof. Let α, β be arbitrary ordinals and do transfinite induction on γ.

- Suppose $\gamma = 0$. Then we have $\alpha^{\beta+0} = \alpha^\beta$. And by the first lemma we know that $\alpha^\beta = 0 + \alpha^\beta$. So then we see that $0 + \alpha^\beta = \alpha^\beta \cdot 0 + \alpha^\beta = \alpha^\beta \cdot s(0) = \alpha^\beta \cdot 1 = \alpha^\beta \cdot \alpha^0$.
- Suppose it holds for γ, so $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$. Then we have $\alpha^{\beta+s(\gamma)} = \alpha^{s(\beta+\gamma)} = \alpha^{\beta+\gamma} \cdot \alpha =_{IH} (\alpha^\beta \cdot \alpha^\gamma) \cdot \alpha$. So by the second lemma we see that $(\alpha^\beta \cdot \alpha^\gamma) \cdot \alpha = \alpha^\beta \cdot (\alpha^\gamma \cdot \alpha) = \alpha^\beta \cdot \alpha^{s(\gamma)}$.
- Now suppose that γ is a limit ordinal and for all $\delta < \gamma$ we have $\alpha^{\beta+\delta} = \alpha^\beta \cdot \alpha^\delta$. Then we have $\alpha^{\beta+\gamma} = \bigcup\{\alpha^\eta; \eta \in \beta + \gamma\} = \bigcup\{\alpha^{\beta+\delta}; \delta \in \gamma\} =_{IH} \bigcup\{\alpha^\beta \cdot \alpha^\delta; \delta \in \gamma\} = \alpha^\beta \cdot \alpha^\gamma$.

□

c)

Proposition. *For all α, β, γ ordinals, we have if $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$.*

Proof. Let α, β be arbitrary ordinals and do transfinite induction on γ.

- Suppose $\gamma = 0$. Then we have $\alpha + 0 = \alpha$ and $\beta + 0 = \beta$. So by our assumption it directly follows that $\alpha + 0 \leq \beta + 0$.
- Suppose it holds for γ, so we have $\alpha + \gamma \leq \beta + \gamma$. Then $\alpha + s(\gamma) = s(\alpha + \gamma)$ and $\beta + s(\gamma) = s(\beta + \gamma)$. So as s is order preserving we see that it follows from the induction hypothesis that $\alpha + s(\gamma) \leq \beta + s(\gamma)$.

- Now suppose that γ is a limit ordinal and that for all $\delta < \gamma$ we have that if $\alpha+\delta \leq \beta+\delta$. Then $\alpha+\gamma = \bigcup\{\alpha+\delta; \delta \in \gamma\}$ and $\beta+\gamma = \bigcup\{\beta+\delta; \delta \in \gamma\}$. So if we can show that the former is a subset of the latter we have the needed result.

  So suppose that $x \in \bigcup\{\alpha+\delta; \delta \in \gamma\}$, then x is of the form $\alpha+\delta$ for some $\delta \in \gamma$. But then by the induction hypothesis we have $\alpha+\delta \leq \beta+\delta$ and thus $\alpha+\delta \subseteq \beta+\delta$. If $\alpha+\delta = \beta+\delta$, then clearly $x = \alpha+\delta \in \bigcup\{\beta+\delta; \delta \in \gamma\}$. So suppose $\alpha+\delta \subset \beta+\delta$. Then by lemma 2.11 iii) we have $\alpha+\delta \in \beta+\delta$. So then $\alpha+\delta \in \bigcup\{\beta+\delta; \delta \in \gamma\}$ by definition of union. So indeed $\alpha+\gamma \subseteq \beta+\gamma$ and thus $\alpha+\delta \leq \beta+\gamma$.

$\square$

Now if we consider the strict version of this statement we get:

If $\alpha < \beta$, then $\alpha+\gamma < \beta+\gamma$.

But if we let $\alpha = 0, \beta = 1$ and $\gamma = \omega$. Then we see that clearly $\alpha < \beta$. But as $\alpha+\gamma = \omega$ and $\beta+\gamma = \omega$. Hence we do not have $\alpha+\gamma < \beta+\gamma$.

d)

Proposition. *For α, β, γ ordinals, if $\alpha < \beta$, then $\gamma+\alpha < \gamma+\beta$.*

Proof. Let α, γ be arbitrary ordinals and do transfinite induction on β.

- Suppose $\beta = 0$. Then $\alpha < 0$, which cannot be the case. So $\beta \neq 0$.
- Now suppose it holds for β, so if $\alpha < \beta$ then $\gamma+\alpha < \gamma+\beta$. Assume that $\alpha < s(\beta)$, then we have $\alpha = \beta$ or $\alpha < \beta$.
  - $\alpha = \beta$: Then $\gamma+\alpha = \gamma+\beta < s(\gamma+\beta) = \gamma+s(\beta)$ as each ordinal is strictly smaller than its successor.
  - $\alpha < \beta$: Then by the induction hypothesis we have that $\gamma+\alpha < \gamma+\beta$. But again $\gamma+\beta < \gamma+s(\beta)$. And thus $\gamma+\alpha < \gamma+s(\beta)$.
- Now suppose that β is a limit ordinal and for all $\delta < \beta$ we have that if $\alpha < \delta$, then $\gamma+\alpha < \gamma+\delta$. So then we see that as $\alpha < \beta$ we get that there is a δ such that $\alpha < \delta < \beta$. So then by the induction hypothesis $\gamma+\alpha < \gamma+\delta \leq \bigcup\{\gamma+\delta; \delta \in \beta\} = \gamma+\beta$.

$\square$

e)

Proposition. *For α, β, γ ordinals, if $\alpha \leq \beta$, then $\alpha \cdot \gamma \leq \beta \cdot \gamma$.*

Proof. Let α, β be arbitrary ordinals and do transfinite induction on γ.

- Suppose $\gamma = 0$. Now $\alpha \cdot 0 = 0 = \beta \cdot 0$. So clearly $\alpha \cdot 0 \leq \beta \cdot 0$.

- Now suppose it holds for γ, so $\alpha \cdot \gamma \leq \beta \cdot \gamma$. As $\alpha \leq \beta$, we have two cases:
  - $\alpha = \beta$: Then we see that $\alpha \cdot s(\gamma) = \beta \cdot s(\gamma)$.
  - $\alpha < \beta$: Then by *d*) we see that $\alpha \cdot \gamma + \alpha < \alpha \cdot \gamma + \beta$. And then by the induction hypothesis and *c*) we see $\alpha \cdot \gamma + \beta \leq \beta \cdot \gamma + \beta$. So $\alpha \cdot s(\gamma) = \alpha \cdot \gamma + \alpha \leq \beta \cdot \gamma + \beta = \beta \cdot s(\gamma)$.
  - Suppose that γ is a limit ordinal and that for all $\delta < \gamma$ we have $\alpha < \delta \leq \beta \cdot \delta$. So $\alpha \cdot \gamma = \bigcup\{\alpha \cdot \delta; \delta \in \gamma\}$. And by the induction hypothesis we have $\bigcup\{\alpha \cdot \delta; \delta \in \gamma\} \leq \bigcup\{\beta \cdot \delta; \delta \in \gamma\} = \beta \cdot \gamma$. So $\alpha \cdot \gamma \leq \beta \cdot \gamma$.

□

Now if we consider the strict version of this statement we have

$$\text{if } \alpha < \beta \text{ then } \alpha \cdot \gamma < \beta \cdot \gamma$$

But if we let $\alpha = 1$, $\beta = 2$ and $\gamma = \omega$. Then we see that $\alpha \cdot \omega = 1 \cdot \omega = \bigcup\{1 \cdot n; n \in \omega\} = \omega$. And also $\beta \cdot \gamma = 2 \cdot \omega = \bigcup\{2 \cdot n; n \in \omega\} = \omega$. So clearly $\alpha < \beta$ but $\alpha \cdot \gamma = \beta \cdot \gamma$ and thus not $\alpha \cdot \gamma < \beta \cdot \gamma$.

Proposition. *For all α, β, γ ordinals, if $\alpha < \beta$ and $\gamma \neq 0$, then $\gamma \cdot \alpha < \gamma \cdot \beta$.*

Proof. Let α, γ be arbitrary ordinals such that $\gamma \neq 0$. Now do transfinite induction on β.

- Suppose that $\beta = 0$. Then $\alpha < 0$, which cannot happen. So this case does not occur.
- Now suppose that it holds for β. So if $\alpha < \beta$, then $\gamma \cdot \alpha < \gamma \cdot \beta$. Now suppose that $\alpha < s(\beta)$, then either $\alpha = \beta$ or $\alpha < \beta$:
  - $\alpha = \beta$: Then $\gamma \cdot \alpha = \gamma \cdot \beta < \gamma \cdot s(\beta)$ as $\gamma \neq 0$.
  - $\alpha < \beta$: Then by the induction hypothesis we know that $\gamma \cdot \alpha < \gamma \cdot \beta$. So as $\gamma \cdot s(\beta) = \gamma \cdot \beta + \gamma$ and $\gamma \neq 0$ we have $\gamma \cdot \alpha < \gamma \cdot s(\beta)$.
- Suppose that β is a limit ordinal and for all $\delta \in \beta$ we have, if $\alpha < \delta$, then $\gamma \cdot \alpha < \gamma \cdot \beta$. As $\alpha < \beta$, there is a δ such that $\alpha < \delta < \beta$. So then $\gamma \cdot \alpha < \gamma \cdot \delta \leq \bigcup\{\gamma \cdot \delta; \delta \in \beta\} = \gamma \cdot \beta$. So we see that $\gamma \cdot \alpha < \gamma \cdot \beta$.

□