Homework Sheet #3

MasterMath: Set Theory
2021/22: 1st Semester
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Deadline for Homework Set #3: Monday, 4 October 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(10) Work in Zermelo set theory Z. A set X is called *Dedekind finite* if every injection from X to X is a bijection. If X and Y are sets, the disjoint union of X and Y is defined to be $X \uplus Y := (\{0\} \times X) \cup (\{1\} \times Y)$.

(a) Show that every natural number is Dedekind finite.

(b) Show that for any natural numbers n and m there is a unique natural number k such that k is in bijection with $n \uplus m$.

(11) Work in Zermelo set theory Z and use the *inductive definitions* of addition and multiplication on $\mathbb{N}$. Show that for all natural numbers n, m, and k, the following hold:

(a) $n + (m + k) = (n + m) + k$,

(b) $n + m = m + n$, and

(c) $n(m + k) = n \cdot m + n \cdot k$.

(12) Work in Zermelo set theory Z. Let $(X, <)$ be a strict total order with minimal element $0 \in X$ and let $S : X \to X$ be a function such that for all $x \in X$, we have $x < S(x)$.

(a) A subset $Z \subseteq X$ is called *S-inductive* if $0 \in Z$ and for all $x \in X$, if $x \in Z$, then $S(x) \in Z$.

(b) A subset $Z \subseteq X$ is called *order inductive* if for all $x \in X$, if $\{z \in X\,;\, z < x\} \subseteq Z$, then $x \in Z$.

(c) We say that $(X, \leq, 0, S)$ *satisfies the principle of complete induction* if for every S-inductive set Z, we have that $Z = X$.

(d) We say that $(X, \leq, 0, S)$ *satisfies the principle of order induction* if for every order inductive set Z, we have that $Z = X$.

(e) We say that $(X, \leq, 0, S)$ *satisfies the least number principle* if every non-empty subset $Z \subseteq X$ has a $\leq$-least element.

Show that the principle of complete induction implies the principle of order induction and that the principle of order induction and the least number principle are equivalent. Give an example of a structure that satisfies the principle of order induction, but not the principle of complete induction. Give conditions on S under which all three principles are equivalent.

(13) Work in FST. Let $\mathbf{X}_1 := (X_1, <_1)$ and $\mathbf{X}_2 := (X_2, <_2)$ be two strict total orders (i.e., irreflexive, transitive, and total relations). We define a new structure $\mathbf{X}_1 \oplus \mathbf{X}_2 = (X, <)$, the *order sum* of $\mathbf{X}_1$ and $\mathbf{X}_2$, as follows:

$$X := X_1 \uplus X_2$$
$$(b, x) < (c, x') :\iff (b = 0 \wedge c = 1) \vee (b = c = 0 \wedge x <_1 x') \vee (b = c = 1 \wedge x <_2 x').$$

Show that $\mathbf{X}_1 \oplus \mathbf{X}_2$ is a strict total order and that it is wellfounded if and only if both $\mathbf{X}_1$ and $\mathbf{X}_2$ are wellfounded.