Homework 3

(10) a) Let $Z = \{n \in \mathbb{N} \mid n$ is Dedekind finite$\}$.

- $n = 0 = \emptyset$. Then trivially any mapping $f: \emptyset \to \emptyset$ is both injective and surjective. Thus every injection from 0 to 0 is a bijection and 0 is Dedekind finite. Thus $0 \in Z$.
- Suppose $n \in Z$, so n is Dedekind finite. Thus any injective function $g: n \to n$ is also surjective. Now consider any injective function $f: s(n) \to s(n)$, where $s(n) = n \cup \{n\}$. Suppose it is not surjective, then there are 2 options:

1) $\forall x \in s(n)$ $f(x) \neq n$, so n is not in the image of f. Now restrict the domain of f to n, one obtains a function $f': n \to n$. As f was injective ~~and $\forall x \in s(n)$ $f(x) \neq n$~~ f' is also injective. As n is Dedekind finite, f' is also surjective. Thus $\forall k \in n$ $\exists l \in n$ $f'(l) = k$. Thus, for f to be injective, $f(n) \notin n$ but we also assumed that $\forall x \in s(n)$ $f(x) \neq n$, so also $f(n) \neq n$. Thus $f(n) \notin s(n)$, a contradiction.

2) $\exists \tilde{x} \in s(n)$ $f(\tilde{x}) = n$ and $\forall x \in s(n)$ $f(x) \neq y$ for some $y \in n$. As f is injective, the $\tilde{x} \in s(n)$ s.t. $f(\tilde{x}) = n$ is unique.
Now consider $f': s(n) \to s(n)$ given by
$f'(\tilde{x}) = y$ (where $\tilde{x}$ was s.t. $f(\tilde{x}) = n$)
and $f'(z) = f(z)$ $\forall z \in s(n) \setminus \{\tilde{x}\}$.
As f was injective, so is f', and $\forall x \in s(n)$ $f'(x) \neq n$. But now we have exactly the situation as in 1), which will yield a contradiction.

Thus any injective function $f: s(n) \to s(n)$ is surjective, and $s(n)$ is Dedekind finite.

$\Rightarrow$ By the principle of induction $Z = \mathbb{N}$ and every natural number is Dedekind finite.

b) Let $Z_m = \{n \in \mathbb{N} \mid n \uplus m$ is in bijection with some $k \in \mathbb{N}\}$, where m is some fixed natural number.

- $n = 0 = \emptyset$. Now take $k = m$ and the function $f: k \to n \uplus m$ given by $f(x) = (1, x)$ for all $x \in k$. Clearly this function is bijective, so $0 \in Z_m$.
- Suppose $n \in Z_m$, consider $s(n) = n \cup \{n\}$. We know there exists a $k \in \mathbb{N}$ s.t. $f: k \to n \uplus m$ is a bijection. Now consider $g: s(k) \to s(n) \uplus m$ given by $g(x) = f(x)$ $\forall x \in k$ and $g(k) = (0, n)$. Clearly this function is bijective, so $s(n) \in Z_m$.

$\Rightarrow$ By the principle of induction $Z_m = \mathbb{N}$ for all $m \in \mathbb{N}$. Thus for any natural numbers n, m

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there is a natural number k s.t. k is in bijection with $n \uplus m$.
Now, suppose there are two natural numbers $k, k' \in \mathbb{N}$, distinct ($k \neq k'$), such that both are in bijection with $n \uplus m$ for some $n, m \in \mathbb{N}$.
W.l.o.g. $k' < k$. By assumption, there are bijections $f : k \to n \uplus m$ and $g : n \uplus m \to k'$, thus $f \circ g : k \to k'$ is a bijection. Restrict the domain of $f \circ g$ to k', one obtains a function $h : k' \to k'$ which is injective as $f \circ g$ was injective. By 10 a) k' is Dedekind finite so h is also surjective. Now as $k' < k$, $\exists x \in k$ $x \notin k'$. For $f \circ g$ to be injective, we have $f \circ g(x) \notin k'$, a contradiction.
Thus for any natural numbers n, m there is a unique natural number k s.t. k is in bijection with $n \uplus m$.

11 a) Fix any $n, m \in \mathbb{N}$. Consider
$Z = \{k \in \mathbb{N} \mid n + (m+k) = (n+m)+k\}$.

- Base case: $k = 0$. Using the Grassmann equations:
  $n + (m+0) = n+m$
  $(n+m)+0 = n+m$ ✓
- Inductive step: suppose $k \in Z$, so $n+(m+k) = (n+m)+k$. Now:
  $n+(m+s(k)) = n + s(m+k) = s(n+(m+k))$
  $(n+m)+s(k) = s((n+m)+k)$
  Thus they are equal and $s(k) \in Z$.

By induction and as $n, m \in \mathbb{N}$ were randomly fixed we can conclude that $\forall n,m,k \in \mathbb{N}$ $n + (m+k) = (n+m)+k$, as $Z = \mathbb{N}$.

b) Consider $Z := \{m \in \mathbb{N} \mid Z_m \text{ is inductive}\}$
where $Z_m = \{n \in \mathbb{N} \mid n+m = m+n\}$.

- Base case: Consider Z_0, to prove it is inductive:
  - base case: $0+0 = 0+0 = 0$ so $0 \in Z_0$
  - inductive step: Suppose $n+0 = 0+n = n$ Now
    $0 + s(n) = s(0+n) = s(n)$
    $s(n)+0 = s(n)$
  Thus $Z_0 = \mathbb{N}$ and $0 \in Z$
- Inductive step: I will use two observations:
  1) $n+1 = n+s(0) = s(n+0) = s(n)$
  2) Z_1 is inductive, proof
     - base case: $1+0 = 1 = s(0) = s(0+0) = 0+s(0) = 0+1$
     - Inductive step: Suppose $n+1 = 1+n$. Now
       $1+s(n) = s(1+n) = s(n+1) = s(s(n)) = s(s(n)+0) = s(n)+1$.
     Thus $Z_1 = \mathbb{N}$ and $1 \in Z$

  Now suppose $n+m = m+n$. Now
  $n+s(m) = n+(m+1) \overset{11a}{=} (n+m)+1 \overset{IH}{=} (m+n)+1$
  $\overset{11a}{=} m+(n+1) = m+(1+n) \overset{11a}{=} (m+1)+n$

Thus as Z_0 and Z_1 are inductive, by induction Z_m is inductive $\forall m \in \mathbb{N}$ and $Z = \mathbb{N}$. Thus $n+m = m+n$ $\forall m, n \in \mathbb{N}$

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c) Fix any $n, m \in \mathbb{N}$. Consider $Z = \{k \in \mathbb{N} \mid n(m+k) = nm + nk\}$

- Base case: $k = 0$, using the Grassmann equations!
  $n(m+0) = n(m) = nm$
  $nm + n0 = nm$ ✓
- Inductive step: Suppose $k \in Z$, so $n(m+k) = nm + nk$. Now:
  $nm + n\,s(k) = nm + nk + n$
  $n(m + s(k)) = n(m + (k+1)) \stackrel{\text{ii a}}{=} n((m+k)+1) = n(s(m+k))$
  $= n(m+k) + n \stackrel{IH}{=} nm + nk + n$

Thus by induction $Z = \mathbb{N}$ and as $n, m \in \mathbb{N}$ were randomly fixed we can conclude that $\forall n, m, k \in \mathbb{N}$ $n(m+k) = nm + nk$.

(13) $X_1 := (X_1, <_1)$ $X_2 := (X_2, <_2)$ strict total orders.
$X_1 \oplus X_2 = (X, <)$. $X := X_1 \uplus X_2$
$(b, x) < (c, x') \Leftrightarrow (b=0 \wedge c=1) \vee (b=c=0 \wedge x <_1 x') \vee (b=c=1 \wedge x <_2 x')$

- To show: $X_1 \oplus X_2$ is a strict total order.
  - Irreflexivity: Suppose for contradiction that $(b, x) < (b, x)$ for some $(b, x) \in X$. There are 2 options:
    1) $b = 0$, then it follows that $x <_1 x$, a contradiction as X_1 is a strict total order.
    2) $b = 1$, then $x <_2 x$, a contradiction as X_2 is a strict total order.

    Thus $(b, x) \not< (b, x)$ $\forall (b, x) \in X$.
  - Transitivity: Suppose $(a, x) < (b, x')$ and $(b, x') < (c, x'')$. There are options:
    1) $a = b = c = 0$, then it follows from the transitivity of X_1 that as $x <_1 x'$ and $x' <_1 x''$, $x <_1 x''$, thus $(a, x) < (c, x'')$.
    2) $a = b = c = 1$, by transitivity of X_2, as $x <_2 x'$ and $x' <_2 x''$, $x <_2 x''$ and thus $(a, x) < (c, x'')$.
    3) $a = b \neq c$ or $a \neq b = c$. In both cases $a = 0$ and $c = 1$, thus $(a, x) < (c, x'')$.
  - Totality: Consider $(a, x), (b, x') \in X$. There are 2 options.
    1) If $a \neq b$, w.l.o.g. let $a = 0$. Then $(a, x) < (b, x')$.
    2) If $a = b$, three suboptions:
       - $a = b = 0$, then as X_1 is total $x <_1 x'$ or $x' <_1 x$, or $x' = x$. Thus $(a, x) < (b, x')$ or $(b, x') < (a, x)$ or $(b, x') = (a, x)$.
       - $a = b = 1$, then as X_2 is total $x <_2 x'$ or $x' <_2 x$ or $x = x'$, so $(a, x) < (b, x')$, or $(b, x') < (a, x)$ or $(b, x') = (a, x)$.

  Thus $X_1 \oplus X_2$ is a strict total order.

- To show: X_1, X_2 wellfounded $\Leftrightarrow$ $X_1 \oplus X_2$ wellfounded.
  (wellfounded $\leftrightarrow$ every nonempty subset has a least element)

$\Rightarrow$ Suppose X_1, X_2 are wellfounded. Take any nonempty subset $Z \subseteq X$. Consider $Z_1 = X_1 \cap \{x \mid (0, x) \in Z\} \subseteq X_1$. If it is nonempty, it has a least element x_1, it follows that $(0, x_1)$ is the least element of Z. If Z_1 is empty, then $Z_2 = X_2 \cap \{x \mid (1, x) \in Z\} \subseteq X_2$ is nonempty, thus it has a least element x_2. It follows that $(1, x_2)$ is the least element of Z.
Thus $X_1 \oplus X_2$ is wellfounded.

$\Leftarrow$ Suppose $X_1 \oplus X_2$ is well founded, but for contradiction X_1 is not. Then $\exists\, Z_1 \subseteq X_1$ where $Z_1 \neq \emptyset$ without a least element. Consider $Z = \{(0,x) \mid x \in Z_1\} \subseteq X$ by assumption this has a least element $(0, x_1)$, but this implies x_1 is the least element for Z_1 ↯

The case when X_2 is not well founded is similar.

Exercise 12. *Assume $(X, \leq, 0, S)$ satisfies the principle of complete induction. We recursively define a function $f\colon \mathbb{N} \to X$: $f(\emptyset) = 0$ and $f(n \cup \{n\}) = S(f(n))$. It can be shown by induction on n that $m < n$ implies $f(m) < f(n)$. By totality, the converse is also true. Since f is strictly increasing, it is one-to-one. The set $\operatorname{ran} f$ is S-inductive hence f is onto. We conclude that f is an isomorphism from $(\mathbb{N}, \subseteq, \emptyset, x \mapsto x \cup \{x\})$ to $(X, \leq, 0, S)$. Hence every property true in the former is also true in the latter. In particular, we have $z < S(x)$ if and only if $z < x$ or $z = x$.*

We now prove that $(X, \leq, 0, S)$ satisfies the principle of order induction. Let Z be an order inductive set. We define $Z' = \{x \in X : \forall z < x\, (z \in Z)\}$. Notice that because Z is order inductive, we have $Z' \subseteq Z$. Clearly $0 \in Z'$ because there is no $z < 0$. If $x \in Z'$, then $S(x) \in Z'$. Indeed, if $z < S(x)$ then either $z < x$ or $z = x$. If $z < x$ then $z \in Z$ by the definition of Z' and if $z = x$ then $z \in Z$ since $Z' \subseteq Z$. Hence Z' is S-inductive thus $Z' = X$. Since $Z' \subseteq Z \subseteq X$, we also have $Z = X$ and the principle of order induction is true.

Let us now assume that $(X, \leq, 0, S)$ satisfies the principle of order induction. We show that it satisfies the least number principle. Let Z be a set with no least element. Then clearly Z^c is order inductive because if for all $z < x$, $z \notin Z$, then $x \notin Z$ because otherwise x would be minimal. Hence $Z^c = X$ and Z is empty.

We finally assume that $(X, \leq, 0, S)$ satisfies the least number principle and show that it satisfies the principle of order induction. Let Z be an order inductive set. Assume $Z \neq X$, then there is a minimal $x \in X$ such that $x \notin Z$. But for every $z < x$, we have $z \in Z$ by the minimality of x hence $x \in Z$ since Z is order inductive. That is absurd, thus we must have $Z = X$.