Homework Sheet #2

MasterMath: Set Theory
2021/22: 1st Semester
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Deadline for Homework Set #2: Monday, 27 September 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(5) Find a graph model $\mathbf{G} = (V, E)$ with vertices v and w such that

$$\mathbf{G} \models \forall z(z \in w \leftrightarrow z \subseteq v),$$

(i.e., w is a power set of v) where v has n predecessors, but w has strictly more than 2^n predecessors. (By a result from Lecture II, this graph cannot satisfy all axioms of FST.)

(6) We use the idea of the constructions from *Group Interaction #1*. Only start working on this homework question *after* your group interaction session.

Let $\mathbf{G} = (V, E)$ be any directed graph. If $v \in V$, we write $\mathrm{pred}_{\mathbf{G}}(v) := \{w \in V\,;\, w\ E\ v\}$ for the set of $\mathbf{G}$-predecessors of v. If $Z \subseteq V$, we say that Z is *managed in* $\mathbf{G}$ if there is some $v \in V$ such that $\mathrm{pred}(v) = Z$. Otherwise, we say that Z is *unmanaged in* $\mathbf{G}$.

The directed graph $\mathrm{vN}(\mathbf{G}) := (V^*, E^*)$ is called the *von Neumann augmentation of* $\mathbf{G}$ if V^* consists of all of the vertices of V plus a set of new vertices V^+ such that each new vertex $v \in V^+$ corresponds to exactly one set $Z \subseteq V$ that is unmanaged in $\mathbf{G}$ with $\mathrm{pred}_{\mathrm{vN}(\mathbf{G})}(v) = Z$. Furthermore, for each $v \in V$, $\mathrm{pred}_{\mathbf{G}}(v) = \mathrm{pred}_{\mathrm{vN}(\mathbf{G})}(v)$.

Given any directed graph $\mathbf{G}$, we define by recursion

$$\mathbf{G}_0 := \mathbf{G} \text{ and}$$
$$\mathbf{G}_{n+1} := \mathrm{vN}(\mathbf{G}_n).$$

Write $\mathbf{G}_n := (V_n, E_n)$ and define $V_\infty := \bigcup_{n\in\mathbb{N}} V_n$ and $E_\infty := \bigcup_{n\in\mathbb{N}} E_n$. We call the directed graph $\mathbf{G}_\infty := (V_\infty, E_\infty)$ the *von Neumann closure of* $\mathbf{G}$.

Start with the graph $\mathbf{H}$ consisting of a single vertex with no edges and form its von Neumann closure $\mathbf{H}_\infty$.

Show that $\mathbf{H}_\infty$ is a locally finite graph that satisfies all the axioms of FST.

(7) Work in FST. A formula $\Phi(x, y, p)$ is called *functional* if the following holds: if $\Phi(x, y, p)$ and $\Phi(x, y, p')$, then $p = p'$. If Φ is a functional formula, we write $\Phi(x, y)$ for the unique p such that $\Phi(x, y, p)$.

The formula Φ is called an *ordered pair definition* if it is functional and for all x, x', y, and y', the following holds: $\Phi(x, y) = \Phi(x', y')$ if and only if $x = x'$ and $y = y'$.

In Lecture II, we stated that Kuratowski's formula $\Phi_{\mathrm{K}}(x, y, p) :\iff p = \{\{x\}, \{x, y\}\}$ is an ordered pair definition. (If you have never seen the proof, check that this is correct.)

Are the following two formulas ordered pair definitions?

(a) $\Psi_0(x, y, p) :\iff p = \{\{y\}, \{x, y\}\}$;
(b) $\Psi_1(x, y, p) :\iff p = \{y, y \cup \{x\}\}$.

(8) Work in FST and show that there cannot be a set of all groups.

[*Hint.* For every set x, there is a group whose universe is $\{x\}$.]

(9) Work in Z and prove that $\in$ is a total strict order relation on $\mathbb{N}$ and that $\subseteq$ is a total order relation on $\mathbb{N}$.

[*Note.* Irreflexivity and transitivity of $\in$ will be proved in Lecture III, so you may skip them.]