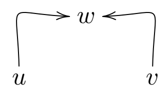
Exercise 5. *The model* **G** *depicted below*



makes $\forall z\,(z \in w \leftrightarrow z \subseteq v)$ *true. However,* v *has* 0 *predecessors and* w *has* 2 *predecessors, which is greater than* 2^0. *The model* **G** *does not satisfy extensionality, so it does not contradict the result from the lecture.*

Set Theory: HW 2.6

Exercise 6. *Start with the graph H consisting of a single vertex with no edges and form its von Neumann closure H_∞. Show that H_∞ is a locally finite graph that satisfies all the axioms of* FST.

Solution. First we note that the von Neumann augmentation of a graph does not yield any new predecessors of existing vertices, just new vertices (with predecessors). Also note that H is finite and if H_n is finite, then it has finitely many unmanaged sets and therefore H_{n+1} is finite. Thus by induction, all H_n for $n \in \mathbb{N}$ are finite.

Locally finite Let $v \in H_\infty$. Then there is some $n \in \mathbb{N}$ such that $v \in H_n$. Since H_n is finite, v has finitely many predecessors in H_n. By the note above, v has fineitely many predecessors in H_∞.

Ext Assume there are distinct $v, w \in H_\infty$ with the same set of predecessors. Then there is some $n \in \mathbb{N}$ such that $v, w \in V_n$ but not $v, w \in V_{n-1}$. If we have $v \notin V_{n-1}$ and $w \notin V_{n-1}$, then v and w were both added in the von Neumann augmentation whereas only one was required, a contradiction. If we have (without loss of generality) that $v \in V_{n-1}$ and $w \notin V_{n-1}$, then the set of predecessors of v is managed in V_{n-1} and therefore, w should not be added in the von Neumann augmentation, a contraction. We conclude that such v, w do not exist.

Pair+Union+Pow For the Axioms of Pairing, Union and Power Set to hold, we need to show that for any vertex (or pair of vertices) some finite set of vertices is managed in H_∞. But there is some $n \in \mathbb{N}$ such that this finite set is contained in H_n and therefore the required set (the pair, union or power set) is present in H_{n+1} and therefore in H_∞.

Sep Let $v \in H_\infty$ and let $\phi(x, \vec{p})$ be any LST formula. Then there is some $n \in \mathbb{N}$ such that $v \in V_n$. Now by the same argument as above, the set $\{z \in v \mid \phi(z, \vec{p})\}$ is a vertex of H_{n+1}, and therefore of H_∞.

△

(7) Observe that $\Psi_0(x,y,p) \iff \Phi_k(y,x;p)$
so for every x,y there is at most one p
that makes it true as Φ_k is functional and
$\Psi_0(x,y) = \Psi_0(x',y') \iff \Phi_k(y,x) = \Phi_k(y',x')$
which is true iff $y=y', x=x'$

Scanned with CamScanner

so Ψ_0 is an ordered pair definition

(b) $\Psi_1(\emptyset, \{\emptyset, \{\emptyset\}\}) = \Psi_1(\{\emptyset\}, \{\emptyset, \{\emptyset\}\})$

so Ψ_1 is not an ordered pair defn.

Exercise 8. *Assume there is a set* U *of all groups. For any set* x*, there is a group with domain* $\{x\}$*, namely* $G = (\ \{x\}\ ,\ \{((x,x),x)\}\ )$*. Assuming pairs are defined using Kuratowski's formula, we have* $x \in \bigcup\bigcup G$*. Hence for all set* x*, we have* $x \in \bigcup\bigcup\bigcup U$*. But according to the axiom of union,* $\bigcup\bigcup\bigcup U$ *is a set and according to the axiom of separation, there is no set of all sets. That is absurd!*

9. We first need a lemma: if $n, m \in \mathbb{N}$ and $n \in m$ then $s(n) \in s(m)$. Use induction. Let $J = \{z \in \mathbb{N} \colon \forall x(x \in z \to s(x) \in s(z))\}$. For the empty set the condition is vacuously true. Suppose $k \in J$ and let $x \in s(k)$. $s(k) = k \cup \{k\}$ so either $x = k$ or $x \in k$. In the former case $s(x) = s(k) \in s(s(k))$. In the latter case, $x \in k \subseteq s(k) \subseteq s(s(k))$. We conclude that J is inductive and therefore $J = \mathbb{N}$. This proves the lemma.

   We begin with totality of $\in$. Let $n \in \mathbb{N}$. We will show by induction that for any $m \in \mathbb{N}$, m is related to n by $=$ or $\in$. Let $Z = \{m \in \mathbb{N} \colon m = n \vee m \in n \vee n \in m\}$. In the lecture we have seen that $\emptyset \in n$ so that $\emptyset \in Z$. Now suppose that $m \in Z$. Distinguish the 3 cases:

   (a) $m = n$. Then $n \in n \cup \{n\} = m \cup \{m\} = s(m)$ so $s(m) \in Z$.

   (b) $n \in m$. Then $n \in m \cup \{m\} = s(m)$ so $s(m) \in Z$.

   (c) $m \in n$. Using the lemma $s(m) \in s(n) = n \cup \{n\}$ so either $s(m) = n$ or $s(m) \in n$. In either case $s(m) \in Z$.

   We conclude that Z is inductive and therefore $Z = \mathbb{N}$. Since n was arbitrary, $\in$ is total.

   Now we show that $\subseteq$ is reflexive, transitive, antisymmetric and total. Let $x, y, z \in \mathbb{N}$.

   Reflexive Every element of x is also in x, so $x \subseteq x$.

   Transitive Suppose $x \subseteq y \subseteq z$. Any element in x is also in y, and also in z, so $x \subseteq z$.

   Antisymmetry Suppose $x \subseteq y$ and $y \subseteq x$. This means that x and y have precisely the same elements. By extensionality they are identical.

   Totality If $x \neq y$, then by totality of $\in$ either $x \in y \vee y \in x$ and because x and y are transitive, either $x \subseteq y$ or $y \subseteq x$.